

How much for a haircut?

Illiquidity, secondary markets, and the value of private equity

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Abstract

Limited partners (LPs) of private equity funds commit to invest with significant uncertainty regarding the timing of capital calls and payoffs and extreme restrictions on liquidity. Secondary markets have emerged which alleviate some of the associated cost. This paper develops a subjective valuation model incorporating these institutional features. Private equity values are sensitive to the discount in secondary market transactions, especially for more risk averse LPs. Model-implied breakeven returns generally exceed empirically observed returns. However, highly risk tolerant LPs may find private equity attractive at portfolio allocations observed in practice, especially if they can access above-average funds and an efficient secondary market.

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Introduction

Institutional investors now routinely allocate 15% to 30% of their portfolios to private equity funds. This capital is committed with extreme restrictions on liquidity. Private equity funds are organized as ten to twelve year partnerships, with investors as limited partners (LPs) and the fund manager as the general partner (GP). Importantly, during the partnership's life there is no option for an LP to redeem their investment directly from the fund. Instead, LPs requiring an early exit from the partnership must turn to the secondary market to sell their stakes, usually at a substantial discount from fund NAV. These haircuts can reach fire sale levels in times of broader market dislocations, such as during the recent global financial crisis. Moreover, even while invested in the fund, LPs face a large degree of uncertainty regarding the timing and magnitude of capital calls for investments in portfolio companies and payoffs from exited portfolio companies. These liquidity arrangements intuitively affect the ex-ante value of commitments to private equity funds, yet to date no academic research attempts to quantify their impact on an LP's required rate of return.

In this paper, we develop a valuation model for commitments to private equity funds from the perspective of a risk averse LP subject to liquidity shocks, such as those brought on by the crisis, which force the LP to seek an early exit from the partnership. In the model, the market is incomplete and private equity cannot be fully spanned by public equity, so present values are not well-defined and an LP's subjective valuation is relevant. The model explicitly incorporates both the secondary market, which we assume the LP accesses when a liquidity shock occurs, and stochastic capital calls and distributions.

The model allows us to shed new light on what is perhaps the central question in assessing private equity as an asset class: What returns are required to compensate for the special liquidity risks and fees LPs face, and how do these compare to empirically observed returns?

We make three main contributions to advance our understanding of these issues. First, we show that both stochastic private equity cash flows and the efficiency of the secondary market have a quantitatively important impact on required returns. Second, we demonstrate that required returns depend on the LP's tolerance for risk, asset allocation, including diversification across public and private equity, and the efficiency of the secondary market. Each of these forces magnifies the impact of the others on required returns. Here, we also examine how an LP's optimal

allocation to private equity depends on the returns it expects. Finally, we show that as a consequence of haircuts in secondary sales, the hold-to-maturity fund returns reported in private equity databases and used in all empirical research are an upper bound on an LP's actual investment experience, and quantify the wedge between reported fund returns and expected LP returns given the possibility of liquidity shocks.

In the spirit of the real options literature, we model the joint evolution of private and public equity asset values using a three-dimensional lattice in discrete time. The lattice allows us to efficiently accommodate a number of path dependencies, including forced secondary sales and stochastic cash flows. The lattice is used to compute the certainty equivalent value of the LP's portfolio at fund inception using backward recursion, exploiting the fact that private equity funds have a finite contracted life.¹

In most of our analyses, we solve for the return of the private equity assets (portfolio companies) required to make the LP's certainty equivalent from a portfolio including private equity equal to that from a benchmark portfolio consisting only of public equity and a risk-free bond. In particular we ask, for a given allocation to private equity, what private equity returns are required to generate the same certainty equivalent as a portfolio consisting of 80% public equity and 20% in a risk-free bond.² From this gross-of-fee breakeven required return on private equity assets we compute the corresponding breakeven net-of-fee LP performance measures commonly used in the academic literature and in practice: the IRR, TVPI, and PME. Naturally, the results depend on the model inputs. We calibrate parameters to the data wherever possible; individual LPs can use our procedure to generate their own benchmarks.

In our first main set of findings, we show that stochastic cash flows and the efficiency of the secondary market have a significant impact on LP valuation, and hence required returns. When secondary markets are relatively inefficient, so haircuts are large, annual required returns on private equity assets are as much as 5 to 10 percentage points higher than when haircuts are small, especially to more risk averse LPs, and a similar difference applies to net-of-fee fund returns. We

¹ Like Sorensen, Wang, and Yang (2014), we model a commitment to a single private equity fund that has a single capital call and a single subsequent distribution.

² A 10 to 20% allocation to fixed income and cash is typical for endowments and affiliated foundations, according to the 2014 NACUBO-Commonfund Study of Endowments. Results are similar using instead a benchmark portfolio consisting of 90% public equity and 10% risk-free bonds.

also show that, in general, required returns are increasing in the LP's risk aversion and allocation to private equity. These effects amplify one another, so that risk tolerant LPs are much less sensitive to the magnitude of secondary market haircuts. Required returns increase with the allocation to private equity much less quickly for risk tolerant LPs.

An important contribution of the model is to provide a new benchmark for assessing the empirical performance of private equity funds. Several recent papers show that, net-of-fee, buyout funds have on average outperformed public equities by 20%-30% over the life of the fund (Kaplan-Schoar (2005) PME of 1.2-1.3), while venture capital funds have roughly equaled public equity after 2000 (see Harris et al. (2014) for a summary). For most parameter values we consider, model-implied breakeven returns exceed these empirical returns, especially for more risk averse LPs. These results indicate that for many LPs the returns they receive in practice are insufficient to compensate for the risks of private equity.

At the same time, highly risk tolerant LPs may find private equity attractive at the 20%-40% portfolio allocations observed in practice, especially if they can access above-average funds and an efficient secondary market. In these cases, model-implied breakeven returns are close to empirical estimates of performance. Importantly, this result occurs only when the risk aversion parameter is around 1.5, which is highly risk tolerant. Even the most risk averse LP we consider, with a risk aversion parameter of 3.5, is considerably more risk tolerant than the marginal public equity market investor, which is often estimated to be in the range of 5-10 (see Mehra and Prescott (1985)). Thus our results suggest that private equity is appropriate only for a highly risk tolerant subset of public equity investors.

The model also allows us to quantify the gap between reported fund returns and LP realized returns. Returns reported in all private equity databases are hold-to-maturity fund returns and overstate the investment experience of an LP subject to liquidity shocks that result in secondary sales. We estimate that this gap is about 3-5 percentage points of IRR per year when the secondary market is relatively inefficient, and falls to 1.5-3 percentage points as efficiency improves. All else equal, the gap is higher when fund performance is better, suggesting that it is the best-performing funds whose reported returns are the most upward-biased estimate of an LP's investment experience.

Next, we consider the impact of fees. A perennial debate in private equity is whether observed fees are excessive. One way to shed light on this issue is to ask what fraction of the gross value generated by the fund accrues to the GP through fees. We find that moving from a hypothetical zero-fee fund to one with the typical 2% per year management fee plus 20% carried interest increases the breakeven private equity asset return by about 5-6 percentage points, or about 50%. Thus in our model GP fees capture about one-third of the gross returns generated by the fund.

Our last analysis solves for the optimal allocation to public and private equity in a three-asset portfolio that includes a risk-free bond. We consider a relatively efficient and a relatively inefficient secondary market, and vary investor risk aversion. Our results indicate that the relatively high allocations to private equity made by large endowments (e.g., Yale had a 35% allocation in 2013) are likely to be optimal to the extent that such institutions are relatively risk tolerant and able to access or select better funds. However, even then the efficiency of the secondary market has a substantial impact on the optimal allocation to private equity. When calibrated to venture capital, for example, the optimal allocation for a risk tolerant investor is 37% with an efficient secondary market but only 27% otherwise. For a risk averse investor, the optimal allocation drops from a much smaller 11% all the way to 6%.

Although these optimal allocations are predicated on expected returns that exceed empirically observed returns, it is possible that LPs expect higher returns than they have historically received. To the extent this is true and as seems likely LP risk tolerance is correlated with size, our model predicts exactly the pattern of allocations seen in practice, where large institutions have private equity allocations in the 30%-40% range, compared to less than 20% for smaller institutions (see NACUBO, 2015).

This paper contributes to several strands of the private equity literature. A key contribution of this paper is to help interpret empirical work on the performance of private equity funds. Robinson and Sensoy (2013), Harris, Jenkinson, and Kaplan (2014), Higson and Stucke (2014), and Phalippou (2013) all use recent data and provide similar estimates of the performance of private equity funds.

Although there is a general consensus on what the performance of private equity funds has been, there is no clear agreement on whether this performance is sufficient to appropriately compensate LPs. Our work adds to a nascent literature attempting to shed light on this question.

In an important contribution, Sorensen, Wang, and Yang (2014) develop a method to value private equity using European option-pricing techniques in continuous time, which requires them to abstract from the secondary market and stochastic capital calls and distributions. No prior work incorporates these key institutional features, which are naturally suited to the real-options framework adopted here and, as we show below, have a quantitatively important impact on the ex-ante value of private equity commitments and, equivalently, breakeven required returns. In addition, despite the fact that institutional investors differ markedly in their allocations to private equity, no prior work examines how breakeven returns vary with the magnitude of these allocations or provides guidance on optimal allocations.

This paper is also related to recent theoretical work in asset pricing analyzing private equity performance measures, particularly the PME. Korteweg and Nagel (2013) and Sorensen and Jagannathan (2013) show that the PME is an appropriate risk-adjusted return for an investor with log utility if the public market portfolio spans the risk of private equity. In this case, the relevant PME benchmark is one. An implication of this paper is that given the liquidity risks of private equity, the appropriate PME benchmark is often substantially greater than one, especially for more risk averse LPs.

The rest of the paper is organized as follows. In Section I we provide some relevant institutional detail. Section II explains our procedure for computing an LP's subjective valuation of private equity investment. Section III discusses base case parameter selection. Section IV presents the results, and Section V concludes.

I. Investments in Private Equity Funds – The Institutional Detail

Private equity funds are generally organized as limited liability partnerships (LLPs) with a contracted life of ten years. The fund is managed by a private equity firm such as Blackstone, which serves as the general partner (GP) of the partnership. Investors in the fund, typically large institutions, are the limited partners (LPs). LPs are passive investors in the fund. At fund inception, LPs commit to provide capital for management fees and for the fund to make investments in portfolio companies. These commitments are not transferred to the GP immediately. Instead, the GP calls capital for investments at its discretion when it identifies investment opportunities.

Similarly, LPs receive cash distributions when the GP chooses to exit portfolio companies through an IPO, acquisition, or liquidation.

As noted above, LPs typically cannot redeem their stakes directly from the GP, in contrast to other forms of delegated asset management such as mutual funds and hedge funds. Liquidity restrictions on LPs are a consequence of the nature of the underlying portfolio company investments. By definition, private equity securities are those for which there is not an organized exchange, and the private equity model is to hold investments for a period of years in an effort to add value, typically through financial or operational engineering. Forced sales of these securities to meet LP redemption demands would impose large transactions costs on the fund and other LPs as well as eliminate the opportunity to add value. Moreover, capital calls and distributions are at the GP's discretion, and hence stochastic from the LP's perspective, because presumably the GP has greater ability than the LP to select and time portfolio company investments and exits. If this were not the case there would be little reason for an LP to invest in a private equity fund.

In response to the illiquidity of LP stakes, secondary markets have emerged which allow an LP to sell their stake to other institutions. The efficiency of these markets can be measured by the discount or haircut a selling LP must accept relative to fair value. Empirical evidence on private equity secondary transactions is scarce, however Kleymenova et al. (2012) report an average bid haircut of 25.2% over the 2003-2010 period. They also report considerable countercyclical variation, with haircuts averaging 8.7% in 2006 and 56.7% in 2009.⁶

GPs are compensated with a management fee and a share of the profits earned by the fund, called carried interest. The most common management fee is 2% of committed capital per year, so that an LP who commits \$100 million pays a total of \$20 million in management fees over a ten-year horizon, leaving \$80 million for investments in portfolio companies. The most common carried interest arrangement is 20% of profits, with LPs receiving their committed capital back, plus a hurdle rate of 8%, before carried interest is earned. Typically, a catch-up provision specifies that GPs receive 100% of further distributions until they have received 20% of total profits. After that point, additional proceeds are split between LPs and GPs according to the carried interest rate.

⁶ In practice, discounts are computed treating fund NAVs as fair values despite the fact that they are potentially subject to manipulation by the GP. Braun et al. (2014), Brown et al. (2014), and Barber and Yasuda (2014) find that NAVs are generally fair, but are sometimes manipulated when the GP is raising capital for a new fund. To the extent NAVs are fair, haircuts represent the cost of obtaining liquidity.

See Gompers and Lerner (1999), Metrick and Yasuda (2010), and Robinson and Sensoy (2013) for descriptions of different management fee and carried interest arrangements.

There are two main types of private equity funds categorized by the nature of their portfolio companies. Venture capital funds invest in early stage pre-IPO companies and do not use debt. Buyout funds, in contrast, invest in more established companies, including public firms, usually with substantial leverage. Consequently, systematic differences in risk and performance are typically observed, hence empirical analysis is often undertaken and results reported on venture capital funds separately from buyout funds. Henceforth when discussing issues that apply to both we use the general term private equity, as in Section II which presents our valuation model. When presenting numerical results, as in Section IV, we report two sets of analyses with parameters selected to represent either venture capital or buyout funds.

II. The Model

Consider a risk averse LP with initial wealth W_0 and a CRRA utility function for date T wealth W_T given by

$$(1) \quad U(W_T) = W_T^{1-\gamma} / (1-\gamma)$$

where γ measures the LP's risk aversion.⁷ The LP can allocate wealth at date 0 across three assets: a risk-free bond, the public equity market, and a private equity fund managed by a GP. Let x denote the fraction of initial wealth W_0 invested in the private equity fund, y denote the fraction of initial wealth allocated to the public market, and $z = (1 - x - y)$ denote the remaining fraction of initial wealth invested in a risk-free bond, with $0 \leq x, y, z \leq 1$. Denote corresponding dollar amounts as X_0 , Y_0 , and Z_0 , respectively.

II.A. Preliminaries

II.A.1. Private Equity Funds

The private equity fund has a maximum investment life of T , though the GP may sell or “liquidate” fund assets and distribute proceeds earlier. The LP's participation in the private equity

⁷ An alternative specification is the utility of lifetime periodic consumption, for which an investment in private equity requires a temporary commitment of capital. See Sorensen et al. (2014).

fund involves a quantity of capital fixed at date 0, known as committed capital, which is itself split into two categories. The first is investment capital, denoted by I_0 , which the LP pays to the GP once suitable assets are identified and the investment capital is called by the GP. The second are management fees, which are a fixed percentage m of committed capital and are paid at the end of every period with the first payment on date 1 and the last payment on the liquidation date. To simplify analysis of fund performance we collapse this arrangement into a single cash outflow by the LP at date 0 and a single cash inflow on the liquidation date. In particular, we assume that both investment capital and the present value of management fees for the full horizon T are set aside at date 0 in an interest bearing account earning the risk-free rate r_f . This avoids uncertainty in the ability of the LP to supply investment capital when called.⁸ Interest earned on the investment capital prior to the call date is retained by the LP in the risk-free account. Management fees are paid at the end of each period out of the account. On the liquidation date, the single cash inflow is defined as the distribution from the GP and the remaining balance in the risk-free account.

Private equity funds often use leverage to increase the scale of asset purchases. We assume the GP issues zero-coupon debt in the amount of D_0 when investment capital is called and assets are purchased so that fund assets are initially worth $A_0 = I_0 + D_0$. The leverage ratio is defined as D_0/I_0 . We assume that debt is effectively risk free, which can be motivated by buyout firms' incentives to implicitly guarantee debt with other assets to maintain the ability to borrow in the future, and hence the yield on fund debt is r_f .⁹

In practice the GP makes multiple capital calls, invests in a portfolio of assets, and distributes proceeds from asset sales as they occur. Like Sorensen et al. (2014), we assume for simplicity a single capital call and a single liquidation date, on which proceeds from asset sales are distributed among debt investors, LPs, and the GP taking into account the seniority of debt investor claims, a hurdle rate for fund performance, and the carried interest earned by the GP. However, unlike prior work, we allow both capital calls and distributions to be stochastic. We

⁸ This uncertainty can be important in practice when, for example, and LP makes multiple commitments in anticipation of a sequence of calls, and the calls instead cluster.

⁹ In practice the debt is not risk-free and would yield something above the risk-free rate. However, given our assumptions regarding expected private equity asset expected returns, the probability that bondholders are not repaid in full has an inconsequential impact on realized bond returns.

denote the share of asset sales flowing to LPs and GPs as $LP(A_t)$ and $GP(A_t)$, respectively. See the Appendix for details on the “waterfall” calculation of these shares.

II.A.2. Joint Evolution of Public and Private Equity

To account for the correlation between the returns of public and private equity, and to allow for other links between the two markets, we need to model their joint dynamics. We assume that the returns of public (U) and private (V) equity are governed by a bivariate normal distribution over each fraction Δt of a year:

$$(2) \quad r_U, r_V \sim BVN(\mu_U \Delta t, \mu_V \Delta t, \sigma_U^2 \Delta t, \sigma_V^2 \Delta t, \rho),$$

where μ and σ are the annual mean and volatility of returns, respectively, and ρ is the correlation between public and private equity returns. In some parts of the analysis we consider stand-alone investments in public or private equity, and in those cases we use the corresponding marginal distribution. Our focus is on the LP’s ability to exit the private equity fund on a secondary market prior to the liquidation date, and for this reason our valuation framework employs a discrete-time numerical technique. Over each period, we allow the value of public and private equity to either increase, stay the same, or decrease, with probabilities and magnitudes chosen to match the distributional assumption in (2). In other words, we calibrate a trinomial distribution to match each assumed normal distribution. Discretizing return distributions in this way is a standard technique in the literature on valuing payoffs with features resembling American options, which arise here because of both secondary sales and stochastic capital calls and distributions.

Figure 1 illustrates the possible paths the value of private equity investment capital can take over the T periods in the life of the fund. Capital committed at date 0 may be called immediately and invested by the GP, in which case its value will either increase, stay the same, or decrease each period. If the investment capital is not called its value remains unchanged until the beginning of the next period. Figure 2 illustrates the possible paths the investment in both public and private equity can take. Figure 2A describes the three dimensions: time and the values of public and private equity. We assume that the investment in public equity occurs immediately upon the allocation decision at date 0. After the capital committed to the private equity fund is called there are nine joint outcomes possible at each point as shown in Figure 2B. As described in the Appendix and corresponding Table A1, we determine the nine branch probabilities by

separating the public and private equity value changes. The public equity evolves according to its marginal distribution, and then the private equity evolves according to its conditional distribution, where the conditioning information is whether the public equity increased, stayed the same or decreased. Figure 2C shows this decomposition.

We achieve changes in asset values via multiplicative factors. From one date to the next, the LP's stake in the public market can increase by a factor e^u , decrease by a factor e^{-u} or stay the same. Once private capital is called and invested by the GP, the value of the purchased assets will increase by a factor e^v , decrease by a factor e^{-v} or stay the same. When analyzing changes in either public or private markets in isolation, we will need the trinomial probabilities for changes in the value of the relevant market according to its marginal distribution. For these, we denote the probabilities of an increase, no change, and a decrease as p_1 , p_0 , and p_{-1} , respectively. In general, we will need the joint probabilities for changes in the values of public and private markets. For these we denote the probability of an increase in both markets as $p_{1,1}$, with other branches defined similarly with the first subscript indicating the change in the public market.

II.A.3. Liquidity Shocks and the Secondary Market

We assume that LPs sell their stakes on the secondary market only in response to a shock to their liquidity needs. At each point in time, we assume there exists the probability ω that a liquidity shock occurs the following period. If a liquidity shock does occur, we assume that the LP sells all risky assets on the secondary market and invests fully in the risk-free bond, i.e., a flight to safety.

In addition to forced sales, the model can accommodate voluntary early exercise by which the LP can decide at each node whether to exercise their real option to transact on the secondary market or continue to hold the position in the private equity fund for an additional period. Early exercise may be optimal, for example, when liquidity shocks are somewhat predictable by the LP, and if haircuts widen substantially when liquidity shocks occur. We find that in our model LP's do not generally exercise early unless the probability of a liquidity shock is quite large. Consequently, for clarity, we omit this possibility in our analyses.

The probability of a liquidity shock is likely related to the performance of risky assets. We set the probability of a liquidity shock at a given node equal to a base rate of 1% per quarter that increases by 1% for every net decrease in public equity asset value. We assume that if the LP sells

their stake in the private equity fund in a secondary market the sale will require the LP to accept a discount or haircut. Empirically, average haircuts vary over time, reflecting the overall appetite for alternative assets and suggesting that the size of the haircut has a systematic component. One might imagine an idiosyncratic component as well, if for example counterparties can prey on LPs that have experienced a liquidity shock.¹⁰ We could make the size of the haircut a function of asset values but for simplicity assume that the haircut can take one of two values. The discount then is a multiplicative factor that is either in a high or low state, H^H or H^L , where H is a haircut with $0 < H^H < H^L < 1$. We denote the probability of each state occurring as p^H or p^L , which we set as a non-linear function of cumulative public equity returns, with a larger (smaller) haircut much more likely when public equity falls (rises) in value. In particular, we set the probability of a large haircut equal to $0.5 - \text{sign}(i) \sqrt{|i|/T}$ where i is the number of net increases, when positive, or decreases, when negative, in public equity at a particular node. See Figure 3 for an illustration of these probabilities. With these assumptions, the size of the haircut conditional on a liquidity shock does not depend on asset values but, because liquidity shocks are more likely when asset values decline, unconditional expected haircuts are larger when asset values are low.

To operationalize these haircuts, suppose that the LP approaches the secondary market at some date t after the call date, and fund asset levels are worth A_t at that time. We assume that the market price for the LP's stake in the fund would be a fraction H of the payoff that would accrue to the LP if the GP were to immediately sell the fund assets, i.e., the LP's payoff after bondholders and the GP's carried interest were paid. Similarly, if the LP approaches the secondary market at some date t prior to or on the call date, the market price for the LP's stake would be a fraction H of the investment capital. In all cases, the LP would retain the current value of interest earned on the deposit of investment capital. With regards to management fees, we assume that the investor who buys from the LP on the secondary market is willing to assume responsibility for paying them. However, the new LP will only pay a maximum of the same percentage management fee as the original LP. If the new LP buys at a price less than the original LP's investment capital, the original LP must make a transfer payment to the new LP equal to the time t value of the difference between

¹⁰ Ramadorai (2012) documents wider haircuts for hedge funds in distress.

the original management fees (which the GP still expects to receive) and those consistent with the same percentage management fee applied to the lower purchase price.

II.B. Asset Valuation in the Lattice

Let $P_{t,i,j,k}$ denote the LP's subjective valuation of its investment at node (t,i,j) , where t denotes the date, ranging from 0 to T , and i and j denote the number of net increases in public and private equity asset values, respectively. As will become clear, we also need to condition the value on the date on which private capital is called; let k denote a potential call date.

At date 0, the probability of capital being called on a given date t is denoted by π_t . To emphasize the uncertainty over capital calls we assume the earliest capital can be called is $t = 1$, reflecting also the time required for the GP to identify suitable assets. Further, let $K < T-1$ denote the last date on which capital can be called, which is typically defined in a partnership agreement. Since we assume public equity investment occurs at time 0, i ranges from t to $-t$. However, on the private equity dimension, a given node is reachable only if the capital was called early enough. In particular, nodes with $|j| \leq t - k$ are reachable.

Denote the probability that the GP sells fund assets and distributes proceeds at a particular node (t,i,j) , and conditional on call date k , as $\phi(t,i,j,k)$. Since the lattice uses a backward recursion, both investment values $P_{t,i,j,k}$ and liquidation probabilities $\phi(t,i,j,k)$ are conditional on reaching a given node without having yet sold the asset. The probability of an asset sale can be a function of the duration of the private equity investment defined by the call date k and the current date t , as well as the cumulative performance of public or private equity assets defined by the number of net increases given by i or j . Presumably, the GP would not sell fund assets prior to T if the GP's payoff, after bondholders and the LP received its share, were zero, since the GP always has an incentive to delay liquidating assets since this allows him to continue to draw management fees each period. For this reason we set the probability of an asset sale at node (t,i,j) , and conditional on a particular call date k , equal to a function of the GP's payoff given an immediate sale as a percentage of the maximum payoff a GP could achieve which occurs when fund assets reach maximum value at date T conditional on the call date k , i.e.

$$(3) \quad \phi(t,i,j,k) = \sqrt{GP(A_0 e^{vj}) / GP(A_0 e^{v(T-k)})}.$$

We take the square root of the ratio in (3) because the probability of an asset sale prior to T would otherwise generally be extremely low given the size of the maximum GP payoff that occurs in the extreme upper node of the lattice.

We solve for the values $P_{t,i,j,k}$ at each possible node using backward recursion given the finite life of the private equity fund. For each node at the penultimate date $T-1$, the valuation is a function of all the possible LP payoffs from fund liquidation at date T represented by all nodes which emanate from the node at $T-1$, exploiting the fact that the GP is sure to liquidate the private equity portfolio at date T . Similarly, date $T-2$ values are obtained using all possible outcomes at date $T-1$, except now these outcomes include both secondary market transactions as well as the continuation values obtained in the prior step, and so on. Each value P is a certainty equivalent, that is, the value of a risk-free bond that provides the same utility for sure as the expected utility of the investment the following period.¹¹

As mentioned, we first consider the range of possible values at date $T-1$. The LP's subjective valuation at node $(T-1,i,j)$, conditional on call date k , can be expressed as:

$$(4) \quad \frac{1}{1-\gamma} \left((1+r_f) P_{T-1,i,j,k} \right)^{1-\gamma} = \frac{1}{1-\gamma} \left(\sum_{ii=-1}^1 \sum_{jj=-1}^1 p_{ii,jj} \left(LP \left(A_0 e^{v(j+jj)} \right) + Int_T + Y_0 e^{u(i+ii)} + Z_T \right)^{1-\gamma} \right)$$

where the LHS is the certain utility of date T wealth generated by investing the date $T-1$ investment value in a risk-free account and the RHS is the expected utility of random date T wealth. The components of the RHS include: the stake in the private equity fund which generates a payoff $LP(\cdot)$ that is a function of private equity asset values at date T , the time T value of interest income earned by the LP on the deposit of investment capital between date 0 and the capital call, denoted by Int_T , the time T value of public equity denoted by $Y_0 e^{u(i+ii)}$, and the time T value of the risk-free bond, Z_T . Since by assumption the liquidation date has not occurred on or before $T-1$, the LP knows the payoff will occur at date T , hence the liquidation probability is irrelevant. Similarly, since by assumption the last possible call date K occurs prior to $T-1$ the LP knows capital has been called and the probability π is irrelevant. The expectation is formed over the nine branches emanating from node $(T-1,i,j)$ reflecting the possible changes in public and private equity asset

¹¹ Ang and Bollen (2010) use a similar approach to investigate the impact of lockups and notice periods on an LP's subjective valuation of hedge funds.

values between date $T-1$ and date T . At node $(T-1, i, j)$ we can solve explicitly for the portfolio value as:

$$(5) \quad P_{T-1, i, j, k} = (1 + r_f)^{-1} \left(\sum_{ii=-1}^1 \sum_{jj=-1}^1 p_{ii, jj} \left(LP(A_0 e^{v(j+jj)}) + Int_T + Y_0 e^{u(i+ii)} + Z_T \right)^{1-\gamma} \right)^{1/1-\gamma}.$$

Now we begin a backward recursion to date 0, i.e., dates t ranging from $T-2$ to 0. As we iterate backwards through time, we take into account uncertainty regarding liquidation and call dates. There are three possible cases involving the relation between t , k , and K as described next.

First, for $t \geq K$, the call date is known by the LP and it is necessary to compute conditional values for all possible call dates:

$$(6) \quad P_{t, i, j, k} = (1 + r_f)^{-1} \left(\sum_{ii=-1}^1 \sum_{jj=-1}^1 p_{ii, jj} \left(\begin{aligned} & \phi \left(LP(A_0 e^{v(j+jj)}) + Int_{t+1} + Fee_{t+1} + Y_0 e^{u(i+ii)} + Z_{t+1} \right)^{1-\gamma} \\ & + (1-\phi) \omega p^H \left(\begin{aligned} & H^H LP(A_0 e^{v(j+jj)}) + \\ & Int_{t+1} + Fee_{t+1}^* + Y_0 e^{u(i+ii)} + Z_{t+1} \end{aligned} \right)^{1-\gamma} \\ & + (1-\phi) \omega p^L \left(\begin{aligned} & H^L LP(A_0 e^{v(j+jj)}) + \\ & Int_{t+1} + Fee_{t+1}^* + Y_0 e^{u(i+ii)} + Z_{t+1} \end{aligned} \right)^{1-\gamma} \\ & + (1-\phi)(1-\omega) P_{t+1, i+ii, j+jj, k}^{1-\gamma} \end{aligned} \right) \right)^{1/1-\gamma}.$$

There are four possible outcomes at each future node. First, with probability ϕ , the GP liquidates private equity fund assets, and the LP receives their share of the proceeds. In this instance we assume the LP marks its portfolio to market, hence we add to the payoff $LP(\cdot)$: Int_{t+1} the time $t+1$ value of interest earned on the deposit of investment capital between date 0 and the call date, Fee_{t+1} , the time $t+1$ value of capital set aside at time 0 to pay future management fees, as these will no longer be paid and can be withdrawn, the value of public equity, and the value of the risk-free bond. Second, with probability $(1-\phi)\omega p^H$ the GP does not liquidate and a liquidity shock occurs with a large haircut. In this case the LP's payoff from private equity is the assumed secondary market value involving the haircut H^H . The withdrawal of remaining fees is adjusted as necessary to account for a transfer payment to the new LP. Third, with probability $(1-\phi)\omega p^L$, the liquidity shock occurs with a small haircut. Fourth, with probability $(1-\phi)(1-\omega)$ neither a GP liquidation

nor a liquidity shock occurs, and the payoff to the LP is the previously computed continuation value.

Second, for $t = K - 1$, the LP will have observed if a call has occurred on or before t . Thus, for $k \leq t$, the valuation is computed as in (6). If a call has not yet occurred, the LP knows capital will be called on date K since K is the last possible call date. In this situation, the value of the investment is:

$$(7) \quad P_{t,i,0,K} = (1+r_f)^{-1} \left(\sum_{ii=-1}^1 p_{ii} \left(\begin{array}{l} \omega p^H \left(H^H I_0 + Int_{t+1} + Fee_{t+1}^* + Y_0 e^{u(i+ii)} + Z_{t+1} \right)^{1-\gamma} + \\ \omega p^L \left(H^L I_0 + Int_{t+1} + Fee_{t+1}^* + Y_0 e^{u(i+ii)} + Z_{t+1} \right)^{1-\gamma} + \\ (1-\omega) P_{t+1,i+ii,0,K}^{1-\gamma} \end{array} \right) \right)^{1/1-\gamma},$$

where p_{ii} are the branch probabilities following the public market's marginal distribution. Note here the haircut is applied to the investment capital I_0 since it has not yet been deployed by the GP. Consequently, the liquidation probability is irrelevant.

Third, for $0 \leq t < K - 1$, the LP again will have observed if a call has occurred on or before t and for $k \leq t$ investment values are given by (6). If a call has not yet occurred, then the LP can form a subjective valuation by considering two outcomes at each adjacent node on date $t+1$: either capital will be called on that date or it won't. So, for $k = t + 1$,

$$(8) \quad P_{t,i,0,k} = (1+r_f)^{-1} \left(\sum_{ii=-1}^1 p_{ii} \left(\begin{array}{l} \omega p^H \left(H^H I_0 + Int_{t+1} + Fee_{t+1}^* + Y_0 e^{u(i+ii)} + Z_{t+1} \right)^{1-\gamma} + \\ \omega p^L \left(H^L I_0 + Int_{t+1} + Fee_{t+1}^* + Y_0 e^{u(i+ii)} + Z_{t+1} \right)^{1-\gamma} + \\ (1-\omega) \left(\pi_{t+1}^* P_{t+1,i+ii,0,k}^{1-\gamma} + (1-\pi_{t+1}^*) P_{t+1,i+ii,0,k+1}^{1-\gamma} \right) \end{array} \right) \right)^{1/1-\gamma},$$

where π_{t+1}^* is the probability of a capital call on date $t+1$ conditional on capital not yet called prior to that date.¹² So, for dates t prior to the penultimate call date, we compute $t+1$ conditional values. The first t are conditional on calls occurring on dates 1 through t . The last is the value conditional on the capital not yet called, and is denoted as a call date $t+1$. As such, the value $P_{t+1,i+ii,0,k+1}$ in the

¹² Under the assumption that the relative probability of subsequent call dates is not revised over time we compute the conditional probability of a call on date $t + 1$ by dividing the unconditional call probability by the sum of the unconditional call probabilities of all possible future call dates from $t + 1$ through K .

RHS of (8) is the value conditional on a call after date $t+1$ and as the iteration through time continues it recursively incorporates all possible future call dates. At date 0 only one value is computed, $P_{0,0,0,1}$, which is the LP's ex-ante valuation of the portfolio.

III. Calibration

Our base case assumes the private equity fund has a twelve year life which we model with 48 quarterly time steps. With regards to the probability of a capital call in a given quarter, Metrick and Yasuda (2010) report that investment periods are typically five years long and that the bulk of the investment occurs in the first three years. We use the cash flow data studied by Robinson and Sensoy (2013) to compute the historical frequency of asset purchases for buyout and venture capital funds. We truncate the distributions at 20 quarters and rescale the frequencies such that the sum of the quarterly probabilities equals one. Figure 4 shows the two corresponding distributions π for venture capital funds and buyout funds. With regards to the probability of asset sales, Metrick and Yasuda use 20% per year post-investment, although they note that in fact the probability is likely an increasing function of asset values. In our analysis, as described in Section II, we set the probability of an asset sale at a given node equal to the square root of the ratio of the GP's share of the proceeds to the maximum share possible given the call date and the cumulative performance of the asset at the node. Thus the distribution probability is increasing in asset performance, as conjectured by Metrick and Yasuda.

We assume annual returns for public equity have a mean of 10% and volatility of 15%, and the risk-free rate is 4%.¹³ Most of our analysis is based on inferring the break-even level of private equity asset returns. That is, we use the model to back out the required private equity return such that the LP's certainty equivalent matches that of a hypothetical portfolio without private equity. For the volatility of private equity assets, Metrick and Yasuda (2010) use a volatility of 60% for individual buyout investments and 90% for individual firms in venture capital portfolios, based on results in Cochrane (2005). They report on average 24 investments per venture capital fund and 15 per buyout fund, and based on Campbell et al. (2001) assume that the pairwise correlation between

¹³ Naturally all parameters used should be forward-looking estimates but for our purpose we rely on historical summary statistics. These vary with the length of the estimation window. Over the 50-year period between 1964 and 2013, the monthly return of the Fama-French market factor has an annualized mean of 10.9% and volatility of 15.6%.

these investments is 20%. With an equal-weight portfolio, these assumptions imply that the portfolio volatility of buyout funds is 30% and the volatility of venture capital funds is 43%. We then round and assume 30% for buyout funds and 45% for venture capital funds.

For the correlation between public and private equity, we use 0.3 for buyout fund assets and 0.6 for venture capital fund assets. Under these assumptions, the market beta of buyout fund assets is 0.6 (so the levered fund beta is 1.2 at 1X leverage) and of venture capital fund assets is 1.8. These betas correspond closely to empirical estimates of the betas of private equity funds, which are typically around one for (levered) buyout betas and two for venture capital funds.¹⁴

To illustrate the lattice topology with quarterly time steps, we use the base case mean and volatility for annual public equity returns of 10% and 15%, respectively. For venture capital and buyout funds we use asset volatilities of 45% and 30%, respectively, and correlations with public equity of 0.6 and 0.3. For both we use an asset expected return of 20% for this illustration; in our main analyses we vary this input to achieve a breakeven certainty equivalent. Listed in Table A1 in the Appendix are branch probabilities determined to generate a discrete trinomial distribution that matches the mean and volatility of each normal distribution. Our procedure described in Appendix 1 results in quarterly step sizes of 9.8% for public equity, 30.9% for venture capital funds, and 19.6% for buyout funds. In panel A, listed for public equity is the marginal probability of each of the three possible returns and listed for the venture capital fund is the conditional probability of each of the three possible returns given one of the public equity returns. Note that the conditional volatility of venture capital fund asset returns is equal in all cases, since the bivariate normal distribution implies a conditional volatility equal to the unconditional volatility scaled by $\sqrt{1-\rho^2}$. The conditional mean of venture capital fund assets is substantially affected by the return of public equity. The conditional mean following a positive public equity return, for example, is 18.2% on a quarterly basis versus -17.2% following a negative public equity return. Panel B shows the corresponding calculations for buyout funds. Note that joint probabilities are higher along the diagonal for venture capital funds reflecting the higher assumed correlation.

¹⁴ See Cochrane (2005), Korteweg and Sorensen (2010), and Driessen et al. (2010).

The tight match between the discrete distribution of the lattice and the assumed continuous distribution is illustrated in Figure 5, which shows the probability of landing on each node at time $T = 48$ as well as the corresponding probability from the bivariate normal distribution.

We assume that liquidity shocks are rare events with base case probability ω equal to 1% per quarter, consistent with the estimate of a global consumption shock in Nakamura et al. (2013) of 3.7% per annum. We change the probability in the lattice at date t by increasing ω by 1% for every net decrease in public equity asset value between date 0 and t . For nodes with a net increase in asset value we leave ω at 1%. The intuition is that an LP is more likely to suffer a liquidity crisis when its relatively more liquid assets are declining in value.

For GP fees we assume a 2% management fee, a 20% performance fee, and an 8% hurdle rate, all of which are typical as found in Gompers and Lerner (1999), Metrick and Yasuda (2010), and Robinson and Sensoy (2013).

With regards to LP risk aversion, Mehra and Prescott (1985) review the use of γ in prior literature, state that most studies use values between one and two, and argue that the parameter should be restricted to a maximum of ten. More recently, Campbell and Cochrane (1999) set γ equal to two in their seminal work on consumption-based asset pricing. As described in the following section, we find that for a twelve year investment horizon, and our assumptions about the risk-free rate and public equity's expected return and volatility, values for γ between 1.5 and 3.5 generate reasonable optimal asset allocations. Hence, these are the values we use throughout the paper.

Table 1 summarizes the base case assumptions for each of the parameters of the model.

IV. Results

IV.A. Benchmark Portfolio of Public Equity and a Risk-free Bond

In much of our analysis it will be helpful to have a benchmark based on public equity for comparison. Consider an investor with a twelve year investment horizon. Using the same type of trinomial lattice described in Section II, and our base case assumptions about the risk-free rate and the expected return of public equity, we determine the allocation weights across public equity and

risk-free bonds that maximizes the certainty equivalent of the portfolio, over a range of risk aversion levels and public equity volatility. We restrict allocation to public equity to be no greater than 100%. Figure 6 shows the results. For low levels of risk aversion and volatility the investor should invest all wealth in public equity. For volatility of 15%, which is typical of estimates of long-run volatility for the stock market, optimal allocation is 100% for $\gamma = 3$ and below. However, given short-term liquidity needs, all investors hold some fraction of wealth in the risk-free asset. In the 2014 NACUBO study of university endowments, for example, the typical endowment allocates roughly 20% to cash and bonds, hence for our benchmark we use a 20% allocation to the risk-free bond with the remaining 80% in public equity. When considering portfolios including private equity, we maintain the 20% allocation to the risk-free bond.

IV.B. Impact of Stochastic Capital Calls and Distributions

As mentioned previously, prior academic work abstracts from the uncertainty of timing of LP cash flows in private equity funds. In this subsection we focus on uncertainty regarding the dates of capital calls and the return of equity capital by the GP, and show that this uncertainty has a significant impact on LP valuations. To illustrate this point, we assume here that there are no liquidity shocks or secondary sales, and focus on the simpler case in which the LP does not invest in public equities. These simplifications allow us to directly associate changes in assumptions about calls and distributions to resulting changes in private equity valuations.

Consider an LP with a risk aversion parameter γ of 2.5 who invests in a portfolio of 80% private equity and 20% in risk-free bonds. We evaluate the LP's ex-ante certainty equivalent value of this portfolio under different assumptions about private equity cash flow timing, reported in Table 2. We do so for three possible private equity funds: a venture capital fund (Panel A), a "1X" buyout fund with debt equal to 100% of initial equity (Panel B), and a "3X" buyout fund with debt equal to 300% of initial equity (Panel C). As described in Section III, we use an expected return and volatility of public equities of 10% and 15%, respectively, and a volatility of 45% and 30%, respectively, for venture capital and buyout fund assets. The risk-free rate is 4%.

In the benchmark case, reported in the first column of Table 2, capital is called immediately at $t=0$ and the distribution occurs at the end of the fund's twelve year life (date T), consistent with Sorensen et al. (2014). Here, we calibrate the expected return of private equity assets so that the portfolio's value is exactly equal to a portfolio of 80% public equity and 20% risk-free bonds,

hence by construction the certainty equivalent is the same in the first column in all three panels. This certainty equivalent is roughly 161% of LP wealth, far in excess of 100% indicating that public equity is attractive to this LP. The required annual expected asset returns are 20.6% for venture capital funds, 17.3% for 1X buyout funds, and 18.4% for 3X buyout funds. We use these expected returns in all other columns to assure comparability.

In the second column of Table 2 the capital call occurs at the end of 2.5 years instead of at $t=0$, reflecting the time spent by the GP in identifying suitable investments. The distribution date remains fixed but occurs prior to year 12. For venture capital funds the distribution occurs at the end of 11.25 years for a total investment horizon of 8.75 years, and for buyout funds the distribution occurs at the end of 9.25 years for a total investment horizon of 6.75 years. These distribution dates are chosen to match those that arise endogenously in the fourth and fifth columns, as described below, so that we can make informative comparisons. In practice, portfolio company holding periods are indeed substantially shorter than the contracted life of the fund as investments are made by the GP over the first few years of the fund's life and exited when the GP decides to do so. Stromberg (2009) reports typical investment horizons for buyout fund assets in the range of five to seven years; typical holding periods for venture capital assets are likely to be somewhat longer given their early stage. The more realistic shorter investment horizons in columns two through five compared to the first column result in substantially decreased valuations for all types of funds.

In the third column we hold distributions fixed as in the second column, and introduce stochastic capital calls by assuming that capital is called with equal probability each quarter over the first 4.75 years, resulting in an expected delay of 2.5 years. Although the expected call date is the same as in the second column, introducing uncertainty in this way lowers the LP's subjective portfolio value by about 1% of initial wealth, for all fund types.

In the fourth and fifth columns we allow stochastic distributions following the expression in Equation (3), with the timing of capital calls the same as in the second and third columns, respectively. The resulting holding periods are endogenously determined as a consequence of the GP's distribution rule. Here, we see that the LP's value is considerably larger than in the cases with fixed distributions. The GP's rule makes distributions more likely when the private equity asset value is high while the fixed distribution does not. This outweighs the negative effect on

valuation of the pure uncertainty. As before, moving from fixed to stochastic capital calls lowers the LP's value.

Overall, Table 2 shows that stochastic capital calls and distributions have a significant impact on the ex-ante value of private equity funds. Uncertainty in the timing of cash flows lowers the risk averse LP's certainty equivalent, but this is offset in the case of distributions by an endogenous GP decision rule that makes higher-value outcomes more likely.¹⁶

IV.C. Impact of Liquidity Shocks, Secondary Market Discounts, Risk Aversion, and Asset Allocation

In this subsection we show that liquidity shocks and secondary market discounts, as well as the LP's asset allocation and risk aversion, have a significant impact on breakeven private equity returns. We also compare breakeven returns to empirically observed returns. For this and all subsequent analyses, we use the full model described in Sections II and III rather than the simplified case used in Section IV.B. As described in Section IV.A., our benchmark for these analyses is a portfolio of 80% public equity and 20% risk-free bonds.¹⁷ We determine the return on private equity assets (portfolio companies) required such that a portfolio consisting of 20% risk-free bonds, $x\%$ private equity, and the remainder in public equity has a certainty equivalent value equal to that of the benchmark portfolio. Thus, these analyses compute the expected return on private equity assets required to make the LP indifferent to replacing some or all of its allocation to public equity with private equity.

IV.C.1. Performance Measures

Obtaining the required breakeven expected return on private equity assets allows us to compute several other performance measures of interest.

First we compute the corresponding asset alpha. As discussed in Section III, we calibrate venture capital assets to an asset beta of 1.8 and buyout to an asset beta of 0.6. The breakeven asset alpha is equal to the breakeven asset return minus the asset beta multiplied by the expected public

¹⁶ All expected returns are constant over time in the model, so there is no sense in which capital calls can be strategic, e.g. to take advantage of depressed asset prices.

¹⁷ This is a tougher, but more appropriate, benchmark than requiring that the portfolio involving private equity simply have a certainty-equivalent at least equal to the LP's wealth. A benchmark of the LP's wealth would be appropriate if the alternative investment were a portfolio of 100% riskless bonds; as shown in Figure 5 and Table 2, the LP can achieve a higher certainty equivalent by investing in public equities.

equity risk premium, which is 6% given our calibration of a 10% expected return on public equity and a 4% risk-free rate.

Both of the asset return measures are gross of fees and so are not comparable to empirically observed returns reported in private equity databases, which are net of fees to the LP. We next compute net-of-fee performance measures. There are two ways to do so. One is to compute the LP's expected net-of-fee return given the breakeven asset return, taking into account the possibility of liquidity shocks and secondary sales at a haircut. This "LP realized" return is not directly comparable to returns in private equity databases, which are instead hold-to-maturity fund returns. Put differently, the returns reported in private equity databases and used in all empirical research on private equity are the returns that would be earned by an LP who never sells on the secondary market. Because some LPs do in fact sell early at a discount, the reported fund returns are higher than the returns realized in expectation by an LP subject to liquidity shocks resulting in secondary sales.

To compute "reported" LP returns for comparison to empirically observed returns, we ask what the net-of-fee LP return would be, given the breakeven expected return on private equity assets, if the probability of a liquidity shock were zero at all times so that the LP by construction holds the fund to maturity. In brief, we compute the breakeven expected asset return and "realized" LP returns under the possibility of liquidity shocks, then compute "reported" LP returns at the same expected asset return by setting the probability of a liquidity shock equal to zero. In this subsection we report "reported" LP returns to allow comparisons with empirical work; we quantify the wedge between "realized" and "reported" LP returns in Section IV.D. below.

We compute three measures of LP return, consistent with the empirical literature: the IRR, TVPI, and PME. Since different paths through the lattice can still result in different exit dates from the fund, we first compute the IRR on a quarterly basis that sets the present value of the payoff equal to the committed capital. Next, we compute a median IRR taking into account the likelihood of each possible payoff and capital call timing occurring in the lattice.¹⁸ The TVPI is the ratio of the undiscounted sum of LP payoffs to the undiscounted sum of capital calls; in our model there are one of each for each possible outcome in the lattice so the TVPI is the ratio between the two.

¹⁸ We focus on median net-of-fee performance metrics due to the presence of influential outliers at the extremes of the lattice.

As with the IRR, we compute a median TVPI over all possible outcomes in the lattice. The PME is the ratio of the discounted sum of LP payoffs to the discounted sum of capital calls, where the discount rate is the (realized, not expected) gross return on the public equity portfolio from any arbitrary reference date to the date of the cash flow in question. Given our assumptions about cash flows, we compute PMEs by scaling each possible payoff at date t by the future value at t of the committed capital grossed up by the public equity return that occurred over that particular path, and report the median PME over all possible paths.

While we compute and report all three measures, we focus on the PME for comparison to empirically observed returns. Both IRRs and TVPIs are highly sensitive to holding periods, among other well-known problems, and as absolute measures of return vary widely through time in the data because of cohort effects and cyclicalities in average performance. In contrast the PME, as a relative performance measure, is much more stable over time in the data and less subject to cohort effects and differences in holding periods, so inferences are likely to be less noisy.

IV.C.2. Venture Capital Funds

Table 3, Panel A reports the break-even expected annual asset returns in venture capital funds as a function of the size of the haircut parameters H^H and H^L , the allocation x to private equity, and the LP's risk aversion and the size of the haircut parameters H^H and H^L . In all cases the breakeven return decreases in the efficiency of the secondary market as modeled by the haircuts. For the least risk averse LP, moving from the least efficient secondary market ($H^L = 80\%$ and $H^H = 50\%$) to the most efficient ($H^L = 95\%$ and $H^H = 90\%$) decreases the breakeven return by about 2 percentage points per year at all allocations. This difference is one measure of the cost of inefficient secondary markets for venture capital funds. For higher risk aversion, moving to a relatively efficient secondary market makes an even larger difference, especially at higher allocations.

The breakeven return increases in the LP's risk aversion, and this effect is more pronounced at higher allocations and when the secondary market is relatively inefficient. In these cases the return to the venture capital fund has a greater impact on the LP's overall payoff and this matters more when the LP is more risk averse. A unit increase in risk aversion raises the breakeven return by about 3-4 percentage points per year at lower allocations or more efficient secondary markets and by about 5-6 percentage points otherwise.

Finally, the breakeven return is also increasing in the allocation to the venture capital fund, indicating that the risks of the fund outweigh the diversification benefits at higher allocations. This effect is especially pronounced when the secondary market is relatively inefficient and the LP is more risk averse. For the least risk averse LP, moving from a 20% to 80% allocation increases the breakeven return by 5.4 percentage points when the secondary market is relatively efficient and 6.1 percentage points when it is relatively inefficient. For the most risk averse LP, the corresponding increases are 12.2 percentage points and 20.7 percentage points, respectively.

In sum, Panel A of Table 3 shows that the efficiency of the secondary market, the LP's allocation to private equity, and the LP's risk aversion all have a quantitatively important impact on the required breakeven returns of venture capital funds. These effects are also mutually reinforcing, as, for instance, the impact of inefficient secondary markets is greater when risk aversion is high and greater still when the allocation to private equity is also high. All of these patterns are qualitatively the same when examining the other performance measures in the other panels of the table.

Panel B of Table 3 reports breakeven alphas. Perhaps surprisingly, breakeven alphas are close to zero, and even negative, for risk tolerant LPs with modest allocations to private equity, especially when the secondary market is relatively efficient. This finding is testimony to the diversification benefits of private equity for risk tolerant LPs. On the other hand, breakeven alphas rise to a level unlikely to be achieved in practice when risk aversion rises, the allocation to private equity rises, or secondary markets are relatively inefficient.

Panels C, D, and E of Table 3 show "reported" LP performance measures that exclude the impact of secondary sales and so are comparable to those empirically observed, as discussed in the prior subsection. In Panel C, we see that the fund IRR, which is the hold-to-maturity LP annual return, is roughly two-thirds of the annual return on the fund's assets (portfolio companies), reflecting the substantial share of gross fund performance that accrues to the GP through the standard fee structure.

In terms of comparisons to the data, empirical returns to venture capital funds vary widely depending on the time period chosen. Focusing on the PME, Harris et al. (2014) report an average (median) PME of 1.36 (1.02) in their full sample period and 0.91 (0.84) since 2000. The latter time period, which excludes the historically unusual late 1990s, is likely more relevant for LPs today.

Based on these figures, LPs with required breakeven PME close to one will find venture capital attractive.

Panel E of Table 3 shows that this condition is satisfied for a risk tolerant LP, as long as the allocation to private equity is less than about 40%, and especially when the secondary market is efficient. For this LP, the breakeven fund PME is 0.95 and 1.13 at a 20% and 40% allocation, respectively, when the secondary market is relatively inefficient. The corresponding breakeven PMEs are just 0.84 and 0.95 when the secondary market is relatively efficient. The reason a net return just equaling or even slightly below public equities is acceptable to this LP is the diversification benefit of private equity, which is outweighed by the risks as allocations increase.

In contrast, Panel E also shows that for more risk averse LPs, required PME benchmarks significantly exceed empirical observations, especially when the secondary market is relatively inefficient. For instance the most risk averse LP requires a PME of 1.23 even with an efficient secondary market and a 20% allocation to venture capital.

Overall, the analysis of breakeven LP returns suggests that the historical performance of venture capital is sufficient to justify its risks and associated fees, but only for a subset of investors that are highly risk tolerant and only at modest allocations. It is also helpful if the LP is in a position to expect a more efficient secondary market. It is likely that larger LPs and those with greater cash reserves would have greater bargaining power in secondary market transactions. In these cases, the benefits of venture capital investments occur mostly because of diversification across asset classes, and high returns per se are not required. These conclusions are consistent with the observation that experienced, aggressive, and large endowments such as Yale typically have greater allocations to private equity than smaller, more historically conservative endowments. Moreover this allocation gap is typically more pronounced for venture capital funds compared to buyout funds.

IV.C.2. 1X Buyout Funds

Table 4 repeats the analysis for the case of a 1X leveraged buyout fund, meaning that asset purchases are financed with 50% equity and 50% debt. In Panel A, we see that all of the qualitative relations between breakeven asset returns and risk aversion, secondary market efficiency, and the allocation to private equity continue to hold with two exceptions. The first is that the breakeven return is U-shaped in the allocation to private equity for the least risk averse LP, meaning that the

diversification benefits of private equity outweigh the special risks at medium allocations but not at low or high allocations. The second is that when the secondary market is relatively efficient and the private equity allocation is low (20%, column 5), the breakeven return is decreasing in risk aversion. This occurs because a more risk averse LP values diversification more. The same phenomenon did not occur in the venture capital case because venture capital funds have both a higher volatility and a higher correlation with public equities leading to a lower diversification benefit.

Turning to comparisons with empirical returns, Harris et al. (2014) report average and median buyout fund PME's in the range of 1.2-1.3 with little difference between the 2000s and the full sample. Panel E of Table 4 shows that the historical performance observed in Harris et al. (2014) is insufficient when the secondary market is inefficient. However with relatively efficient secondary markets, required PME's are close to empirically observed performance for the most risk tolerant LP at private equity allocations up to 60%, and for the more risk averse LPs at allocations of 20%. These patterns echo those for venture capital in Table 3, although buyout funds are attractive at more parameter combinations. Taken together, the results help explain why many institutional investors have recently pared down their exposure to private equity, and especially venture capital, even while risk tolerant and large endowments such as Yale have remained stalwarts.

IV.C.3. 3X Buyout Funds

Table 5 considers the case of a 3X buyout fund, in which asset purchases are financed with 25% equity and 75% debt. With higher leverage comes higher volatility that weighs against the diversification benefit, and here in no case do we see required returns decreasing in risk aversion. The U-shaped pattern of required returns in the allocation to private equity likewise disappears.

The required PME's in Panel E are almost all higher than those in Table 4, reflecting the greater risk with higher leverage. A 3X buyout fund is only attractive to a highly risk tolerant LP.

IV.C.4. Discussion

Overall, the results in Tables 3-5 suggest a nuanced view of the attractiveness of private equity. The results strongly suggest that private equity is not appropriate for all or even most potential LPs. Instead LPs more likely to avoid disappointment in private equity are those that are

risk tolerant, able to access or select at least average funds, and face a relatively efficient secondary market, perhaps because of a stronger bargaining position owing to deep pockets.

IV.D. The Wedge between Reported Fund Returns and Realized LP Returns

As discussed above, reported fund returns are an upper bound on the returns actually earned by an LP in the fund because they do not take into account the possibility of lower payoffs that result from secondary sales. As a result fund returns are an upward-biased measure of the investment experience of a typical LP.

In this subsection we estimate the wedge between reported fund returns and those that are expected by an LP subject to liquidity shocks resulting in secondary sales. To do so we compute LP realized net-of-fee returns using the breakeven asset returns reported in Tables 3-5 and taking into account the possibility of liquidity shocks and secondary sales.

Table 6 shows the results for venture capital funds. Panel A shows the breakeven reported fund return (IRR) and Panel C the breakeven reported fund PME. These are exactly the same as, respectively, Panels C and E in Table 3, reproduced for convenience. Panels B and D of Table 6 show the corresponding realized LP returns.

Naturally, the wedge between fund reported and LP realized returns is greater when the secondary market is relatively inefficient. In this case the wedge is about 3-5 percentage points of IRR, compared to 1.5-3 percentage points when the secondary market is more efficient. In terms of PME the wedge varies more widely, as a given wedge in IRR results in a larger wedge in PME when the baseline IRR is greater. The PME wedges are somewhat less than would be obtained by cumulating the IRR wedges up over a typical investment holding period, both because of Jensen's inequality and because the liquidity shocks that drive the wedges are more likely when the public equity values have decreased.

In all cases the wedge is increasing in the fund return. An implication of this is that it is the best-performing funds for which the fund return is the most upward-biased measure of the investment experience of an LP. On the other hand, an LP who can strategically sell only poorly performing funds on the secondary market in response to a liquidity shock may realize returns close to the reported fund returns.

Tables 7 and 8 report the results for 1X and 3X buyout funds, respectively. In each case the wedge between fund reported and LP realized IRRs is about the same as for venture capital funds.

IV.E. Impact of Fees

A continuing debate is whether fees charged by private equity funds to their investors are appropriate. One way to gauge the magnitude of fees is to ask how LPs' required asset returns change under different fee structures. Put differently, for a given change in fees, by how much does the gross value generated by the fund need to change to keep LPs indifferent?

Table 9 presents estimates from a portfolio that is 40% private equity, 40% public equity, and 20% risk-free bonds for the LP with $\gamma = 1.5$. The way in which the required return changes with the level of fees does not depend strongly on the efficiency of the secondary market. Increasing management fees results in a greater than one-for-one increase in required asset returns. This is because management fees must be paid even before investments are made by the fund. Carried interest, by contrast, does not have as large an effect. Increasing carried interest from 15% to 25% increases the required asset return by only 0.5-1 percentage points per year.

The final main message from Table 9 is that relative to a hypothetical zero-fee, zero-carry benchmark, the typical 2% management fee plus 20% carried interest structure increases the breakeven asset return by about 50% (not percentage points). In turn, this implies that at typical fee levels private equity GPs capture about one-third of gross performance they generate.

IV.F. Optimal Portfolio Weights

Our last analysis fixes the bond allocation to 20% and then finds the utility-maximizing split between public and private equity. We do this for each type of private equity fund at the most efficient and least efficient secondary market parameters. We assume here that venture capital assets have expected return of 18%, 1X buyout assets have an expected return of 15% per year, and 3X buyout assets have an expected return of 12%. These parameters are chosen, based on the breakeven analysis reported in Tables 3-5, to ensure that even the most risk averse LP will find some allocation to private equity attractive.

Results are displayed in Table 10. In Panel A, for the least risk averse investor, the optimal allocation to venture capital increases from 25% to 37% when secondary markets are made more

efficient. For the most risk averse investor the increase is smaller in absolute terms but larger in percentage terms, rising from 5% to 11%. This result indicates that the efficiency of the secondary market has a substantial effect on the optimal portfolio. Also, the optimal allocation to venture capital is widely varying across risk-aversion levels. Results in Panels B and C for the 1X and 3X buyout funds are qualitatively similar.

Interestingly, as reported in the 2015 NACUBO study, the allocations in the more efficient secondary market case correspond reasonably closely to the average allocation to alternative investments in 2014 for university endowments as a function of the size of the endowment, to the extent that larger endowments are less risk averse. That study reports that endowments over \$1 billion have a 57% allocation to alternatives, while endowments under \$100 million have less than 20%.

VI. Conclusion

This paper presents a valuation framework for private equity investment from the perspective of a risk averse LP, explicitly incorporating institutional features that differentiate private equity from other types of vehicles. In particular, we allow for a distribution of call dates thereby reflecting the uncertainty of how long capital is actually deployed in private equity assets. We also allow for uncertainty surrounding when assets are liquidated by the GP. Most importantly, we model LP exits on a secondary market in response to liquidity shocks. Secondary markets are quickly developing in practice to remedy the severe illiquidity in private equity investment. Up until now, there has been no research that studies the impact of the efficiency of secondary markets on the valuation of private equity stakes.

We find that cash flow uncertainty does matter, both by reducing the average time that capital is deployed by GPs as well as creating additional uncertainty in the distribution of LP payoffs. In our model, we assume that LPs set aside all committed capital at date 0 in an interest-bearing account, hence a distribution of call dates naturally leads to periods of low earning power for LPs. We also find that the efficiency of the secondary market can have a dramatic impact on the attractiveness of private equity investment, in some cases reducing the break-even fund return by 10% per year.

Our model can be used to guide LPs in their ex-ante allocation decision, as well as ex-post analysis of performance. In both cases our procedure requires as an input the LP's assessment of the distribution of call dates as well as the subsequent likelihood of a GP's distribution of fund proceeds. These parameters can be studied using historical data of LP cash flows. The model can also be calibrated to an LP's individual assessment of its susceptibility to liquidity shocks and subsequent need for secondary sales.

We make a number of simplifications to our representation of an LP's decision problem to gain tractability. Relaxing at least two of these is a fruitful direction for future research..

First, for a given private equity investment, we assume that all committed capital is called on a single date, and all fund assets are liquidated on a single subsequent date. This assumption likely overstates the cost of illiquidity, since in practice LPs hold a portfolio of private equity investments, with partial calls and liquidations for each, so that the allocation to private equity in whole is somewhat self-financing. That said, the popular press has documented cases in which the systematic component of calls and distributions has left LPs with significant liquidity crises.

Second, we assume the LP has an investment horizon equal to the contracted life of a single private equity investment. Our valuation model therefore focuses on this relatively short time period. In practice, private equity investors often have a very long investment horizon. For University endowments, for example, one could argue the relevant horizon is infinite given the permanent role that endowments can play in a University's budget. However, it is likely that the efficiency of the secondary market will continue to play a significant role in valuation even at longer horizons, and payoffs at horizons greater than the twelve years studied here will have a relatively modest impact on $t=0$ utility at discount rates appropriate for risky investments.

Appendix

A.1. Waterfall

The computation of claimant shares follows closely that in Sorensen et al. (2014). Suppose that fund assets are worth A_t on the liquidation date t . Distributions to the three types of claimants are defined by three discrete asset levels denoted A_1 , A_2 , and A_3 . The first, A_1 , is the amount due debt investors, i.e.

$$(A1) \quad A_1 = D_0 (1 + r_f)^{(t-k)}.$$

Note that the value of the debt reflects the time from issuance, which is the call date k . We assume that debt investors receive 100% of asset proceeds until they receive A_1 , the full amount they are due. The second, A_2 , is defined by the LP's hurdle rate h applied to the committed capital and all management fees paid to the GP, i.e.

$$(A2) \quad A_2 = A_1 + I_0 (1 + h)^{(t-k)} + (mY_0/r_f) \left(1 - (1 + r_f)^{-t}\right) (1 + h)^t.$$

Note that the hurdle rate is applied to the investment capital only from the call date k , whereas the hurdle rate is applied to all management fees paid beginning on date 1. We assume that, once the debt investors are fully repaid, the LP receives 100% of additional asset proceeds until they receive $A_2 - A_1$.

The third, A_3 , is defined by the GP's carried interest, c , which is a percentage of fund profits, typically 20%, as follows:

$$(A3) \quad c(A_3 - (Y_0 + A_1)) = A_3 - A_2.$$

The left-hand side of (A3) is the GP's contractual carried interest payment and the right-hand side of (4) is the "catch-up" region of asset sales, assuming that the GP receives 100% of proceeds beyond A_2 until its carried interest is paid. The solution to (A3) is $A_3 = (A_2 - c(Y_0 + A_1)) / (1 - c)$. Beyond A_3 , asset sale proceeds are split between the LP and GP with the GP receiving a fraction c of the incremental proceeds.

A.2. Branch Probabilities

When considering private equity (V) investment as a stand-alone investment we assume returns r_v over each period of length Δt are normally distributed

$$(A4) \quad r_v \sim N(\mu_v \Delta t, \sigma_v^2 \Delta t).$$

where μ and σ are annual mean and volatility, respectively. Following Kamrad and Ritchken (1991) we can approximate the continuous normal distribution of r_v using a discrete random variable r_v^a governed by the trinomial distribution used in the lattice. From any node in the lattice, the discrete variable can take on one of three values

$$(A5) \quad r_v^a = \begin{cases} v = \lambda_v \sigma_v \sqrt{\Delta t} & \text{with probability } p_1 \\ 0 & \text{with probability } p_0 \\ -v = -\lambda_v \sigma_v \sqrt{\Delta t} & \text{with probability } p_{-1} \end{cases}$$

where $\lambda_v \geq 1$ scales the familiar Cox, Ross, and Rubenstein (1979) binomial lattice step size. The free parameter λ_v is selected numerically to both ensure all probabilities between 0 and 1 and to keep the three branch probabilities as close to 1/3 as possible to aid the ability of the discrete

distribution to approximate the assumed continuous normal distribution.²² Probabilities are selected so that the mean and variance of the approximating distribution match those of the true distribution for portfolio returns:

$$(A6) \quad \begin{aligned} E(r_v^a) &= (p_1 - p_3)v = \mu_v \Delta t \\ \text{Var}(r_v^a) &= (p_1 + p_3)v^2 - (p_1 - p_3)^2 v^2 = \sigma_v^2 \Delta t. \end{aligned}$$

Using (A6) we can solve for the three branch probabilities as:

$$(A7) \quad \begin{aligned} p_1 &= \frac{1}{2} \left[\frac{\sigma_v^2 \Delta t + \mu_v^2 \Delta t^2}{v^2} + \frac{\mu_v \Delta t}{v} \right] \\ p_0 &= 1 - \frac{\sigma_v^2 \Delta t + \mu_v^2 \Delta t^2}{v^2} \\ p_{-1} &= \frac{1}{2} \left[\frac{\sigma_v^2 \Delta t + \mu_v^2 \Delta t^2}{v^2} - \frac{\mu_v \Delta t}{v} \right]. \end{aligned}$$

When considering a portfolio of both private equity and investment in public markets, we need to model the marginal distribution of public equity (U) as well as the joint distribution of public equity and private equity, consistent with the bivariate normal assumption:

$$(A8) \quad r_U, r_V \sim BVN(\mu_U \Delta t, \mu_V \Delta t, \sigma_U^2 \Delta t, \sigma_V^2 \Delta t, \rho).$$

where ρ is the correlation between public equity and private equity returns.

Boyle (1988) and Kamrad and Ritchken (1991) also make the assumption of bivariate normality in their analysis of options on two variables. In both papers, the authors allow variables to either increase, decrease, or stay the same over any increment of time. However, they restrict the process by assuming that if one variable stays the same, both do. In other words, there are five possible outcomes for the discrete joint distribution. In our analysis of public and private equity, we need to allow public equity values to change while private equity remains unchanged to model the situation between capital commitment and a capital call. Hence we develop our own procedure for establishing the topology of the lattice and the branch probabilities.

Boyle (1988) and Kamrad and Ritchken (1991) solve for all branch probabilities simultaneously by matching the mean and variance of each variable as well as the correlation between the two. In contrast, we employ a much simpler procedure that exploits the conditional distribution of one of the assets.²³

The procedure has two steps. In the first step, trinomial branch probabilities established in (A7) above are selected to match the marginal distribution of public equity. The marginal distribution of public equity is:

$$(A9) \quad r_U \sim N(\mu_U \Delta t, \sigma_U^2 \Delta t).$$

The discrete trinomial distribution that best approximates (A9) is:

$$(A10) \quad r_U^a = \begin{cases} u = \lambda_U \sigma_U \sqrt{\Delta t} & \text{with probability } p_1 \\ 0 & \text{with probability } p_0 \\ -u = -\lambda_U \sigma_U \sqrt{\Delta t} & \text{with probability } p_{-1} \end{cases}$$

²² See Kamrad and Ritchken (1991) for a discussion.

²³ Ho, Stapleton, and Subrahmanyam (1995) also exploit conditional probabilities when valuing options on multiple assets.

where the parameter λ_v is determined numerically as above and the probabilities are defined by (A7) using public equity's mean and volatility.

In the second step, three sets of trinomial branch probabilities are computed to match the conditional distribution of private equity, where the conditioning information is the return experienced by public equity in the first step. For a bivariate normal distribution, the distribution of the return of private equity conditional on the return of public equity is:

$$(A11) \quad r_{v|u} \sim N\left(\mu_v \Delta t + \rho \sigma_v / \sigma_u (r_u - \mu_u \Delta t), \sigma_v^2 (1 - \rho^2) \Delta t\right)$$

Naturally the joint probabilities can then be computed as the product of the marginal and conditional branch probabilities. In our discrete approximation, there are three possible returns for public equity, and hence three possible conditional distributions for private equity, which differ only by their mean. In each of the three cases the conditional variance is $\sigma_v^2 (1 - \rho^2) \Delta t$ and this value is used in place of $\sigma_v^2 \Delta t$ in (A7). Furthermore, the step size in (A5) v now equals $\lambda_v \sigma_v \sqrt{(1 - \rho^2) \Delta t}$ with the free parameter λ_v selected via numerical methods to ensure all conditional probabilities lie between 0 and 1 and are as close to 1/3 as possible to aid convergence properties. We use a single step size for all three conditional distributions to maintain efficient lattice construction.

Case 1: Public equity increases with return $r_u^a = u$. This results in a conditional mean for private equity equal to $\mu_v \Delta t + \rho \sigma_v / \sigma_u (u - \mu_u \Delta t)$, which is used in place of $\mu_v \Delta t$ in (A7).

Case 2: Public equity has no change with return $r_u^a = 0$. This results in a conditional mean for private equity equal to $\mu_v \Delta t - \rho \sigma_v / \sigma_u (\mu_u \Delta t)$, which is used in place of $\mu_v \Delta t$ in (A7).

Case 3: Public equity decreases with return $r_u^a = -u$. This results in a conditional mean for private equity equal to $\mu_v \Delta t - \rho \sigma_v / \sigma_u (u + \mu_u \Delta t)$, which is used in place of $\mu_v \Delta t$ in (A7).

To illustrate, suppose that public equity has annual mean and volatility of 10% and 15%, respectively, whereas private equity has annual mean and volatility of 20% and 30%, and that the two are jointly normal with correlation of either 0.3 or 0.6. These values correspond to quarterly means of 2.5% and 5.0% for public and private equity, respectively, and quarterly volatilities of 7.5% and 15.0%. Table A1 shows that the trinomial distribution for public equity's marginal distribution has a step size u of 9.8% per quarter, with probabilities roughly 45%, 35%, and 20% for an increase, no change, and a decrease. In Panel A, with correlation .3, the conditional probability of private equity has a step size v of 19.6% per quarter. Conditional is 14.3% per quarter, regardless of public equity's change, whereas the conditional mean is 9.4% following an increase in public equity, and lower values otherwise. Panel B shows the joint probabilities for each combination of moves. Panels C and D repeat the calculations for a higher correlation of .6. Note that the joint probabilities in Panel D are higher along the diagonal than in Panel B.

Table A1. Branch Probabilities

Listed are branch probabilities determined to generate discrete trinomial distributions that match the mean and volatility of corresponding normal distributions. In Panel A, public and venture capital fund returns are governed by a bivariate normal distribution, with quarterly means of 2.5% and 5%, respectively, quarterly volatilities of 7.5% and 22.5%, and correlation of 0.6. In Panel B, public and buyout fund returns are governed by a bivariate normal distribution, with annual means of 2.5% and 5%, respectively, volatilities of 7.5% and 15%, and correlation of 0.3. Listed for public equity is the marginal probability of each of the three possible returns from the marginal probability distribution, and listed for both types of private equity is the conditional probability of each of the three possible returns given one of the public equity returns. The quarterly step sizes are $u = 9.8\%$ for public equity, $v = 30.9\%$ for venture capital fund assets, and $v = 19.6\%$ for buyout fund assets.

| | | Panel A. Venture Capital Fund | | | | | | | |
|-------------------------|-------------------------|--------------------------------------|----------|-------------------------|--------|--------|-------------------|--------|--------|
| Public Equity Return | Marginal Probability | Conditional Distribution | | Conditional Probability | | | Joint Probability | | |
| | | μ | σ | v | 0 | $-v$ | v | 0 | $-v$ |
| u | 0.4523 | 18.2% | 18.0% | 0.6371 | 0.3139 | 0.0490 | 0.2882 | 0.1420 | 0.0221 |
| 0 | 0.3502 | 0.5% | 18.0% | 0.1783 | 0.6596 | 0.1621 | 0.0624 | 0.2310 | 0.0568 |
| $-u$ | 0.1974 | -17.2% | 18.0% | 0.0466 | 0.3510 | 0.6024 | 0.0092 | 0.0693 | 0.1189 |

| | | Panel B. Buyout Fund | | | | | | | |
|-------------------------|-------------------------|-----------------------------|----------|-------------------------|--------|--------|-------------------|--------|--------|
| Public Equity Return | Marginal Probability | Conditional Distribution | | Conditional Probability | | | Joint Probability | | |
| | | μ | σ | v | 0 | $-v$ | v | 0 | $-v$ |
| u | 0.4523 | 9.4% | 14.3% | 0.6222 | 0.2352 | 0.1426 | 0.2814 | 0.1064 | 0.0645 |
| 0 | 0.3502 | 3.5% | 14.3% | 0.3728 | 0.4332 | 0.1939 | 0.1306 | 0.1517 | 0.0679 |
| $-u$ | 0.1974 | -2.4% | 14.3% | 0.2139 | 0.4504 | 0.3357 | 0.0422 | 0.0889 | 0.0663 |

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Figure 1. Trinomial Lattice

The lattice shows possible changes in the value of capital committed by a limited partner (LP) at date 0. The capital may remain uninvested, in which case its value stays unchanged, or it may be called and invested, in which case the value can increase multiplicatively by one unit, stay the same, or decrease by one unit over each time period. The fund has finite life of T periods, hence T periods of change are possible if the capital is committed at date 0.

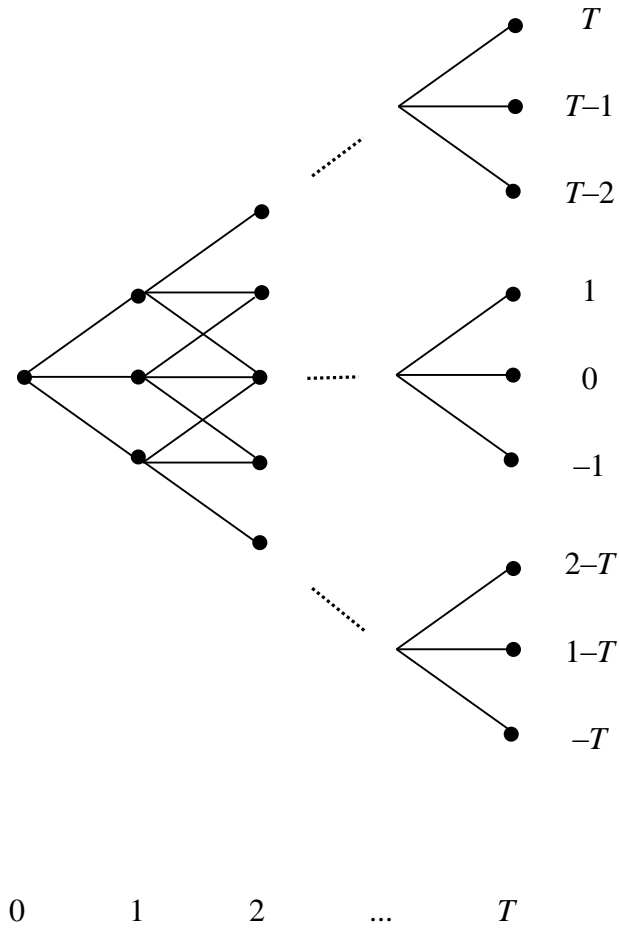


Figure 2. A Bivariate Distribution for Public and Private Equity

Figure 2A shows the three dimensions in the lattice used to model the bivariate distribution for changes in public and private equity. Figure 2B shows the nine possible joint outcomes over each time increment allowing for both public and private equity to increase, decrease, or stay the same. Figure 2C shows the same nine outcomes generated in two stages. In the first, the change in public equity is determined following its marginal distribution. In the second, the change in private equity is determined following its conditional distribution, in which the conditioning information is the change in public equity.

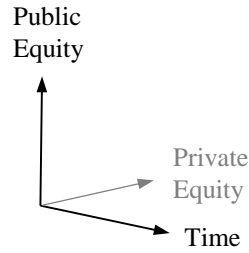


Figure 2A. The Three Dimensions in the Lattice

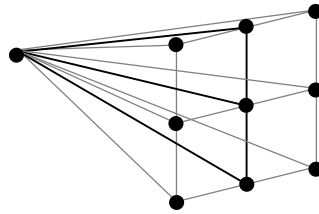


Figure 2B. Possible Outcomes Generated by the Bivariate Distribution of Public and Private Equity

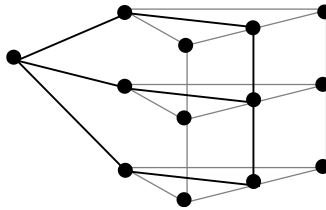


Figure 2C. Possible Outcomes Generated by Public Equity's Marginal Distribution Followed by Private Equity's Conditional Distribution

Figure 3. Probability of a Large Haircut

Depicted is the probability of a large haircut as a function of the cumulative number of net increases or decreases in public equity values.

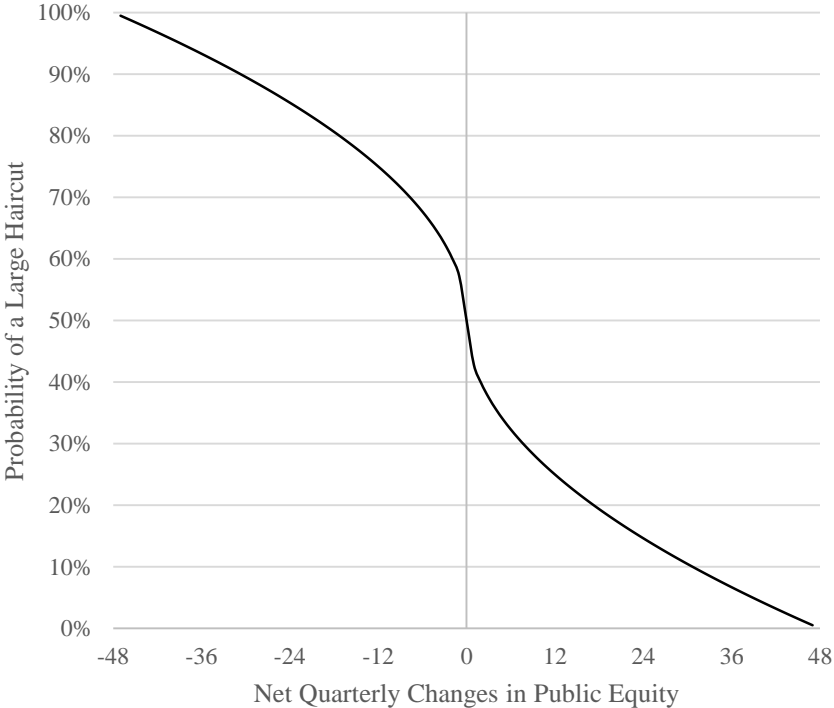


Figure 4. Capital Calls

Depicted for venture capital and buyout funds are the empirical distributions of dates on which the GP calls capital and makes an acquisition.

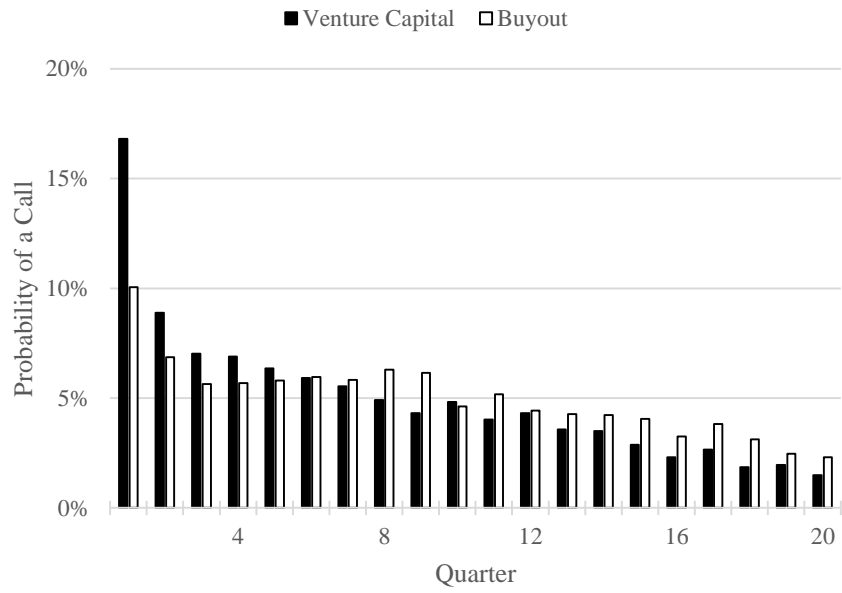


Figure 5. Distribution Comparison

Figure 5A shows the joint probabilities of annualized private equity and public equity returns implied by a 48-step lattice approximating a 12-year investment horizon. Figure 5B shows the analytic probabilities from the underlying bivariate normal distribution.

Figure 5A. Probabilities Implied by the Lattice

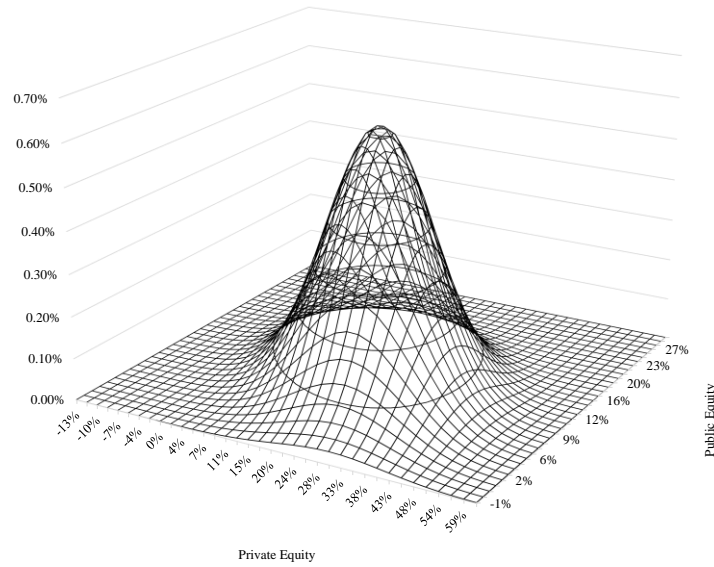


Figure 5B. Probabilities Implied by the Bivariate Normal Distribution

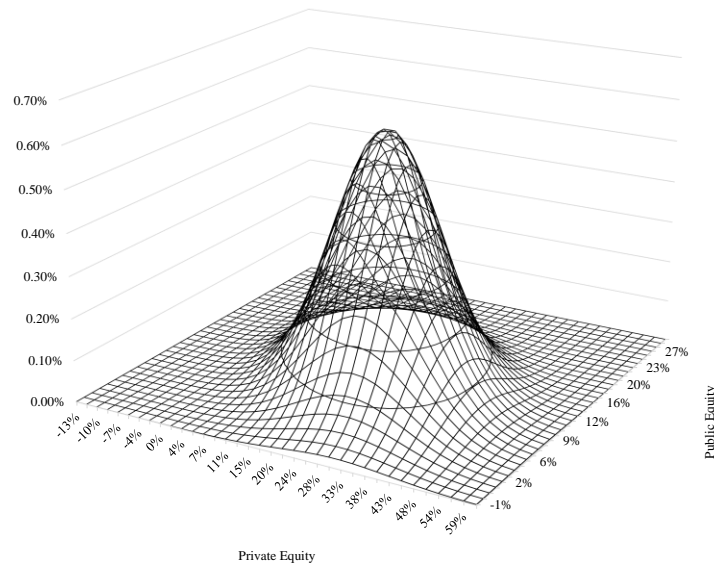


Figure 6. Optimal Allocation between Public Equity and a Risk-free Bond

Depicted is the optimal allocation to public equity with annual expected return of 10%, and volatility levels as listed, for a risk averse investor, with the remainder of wealth invested in a risk-free bond with annual return of 4%. The investor has risk aversion levels as displayed on the horizontal axis. Liquidity shocks can occur at a base rate of 1% per quarter which increases by 1% for every net quarterly decline in public equity values.

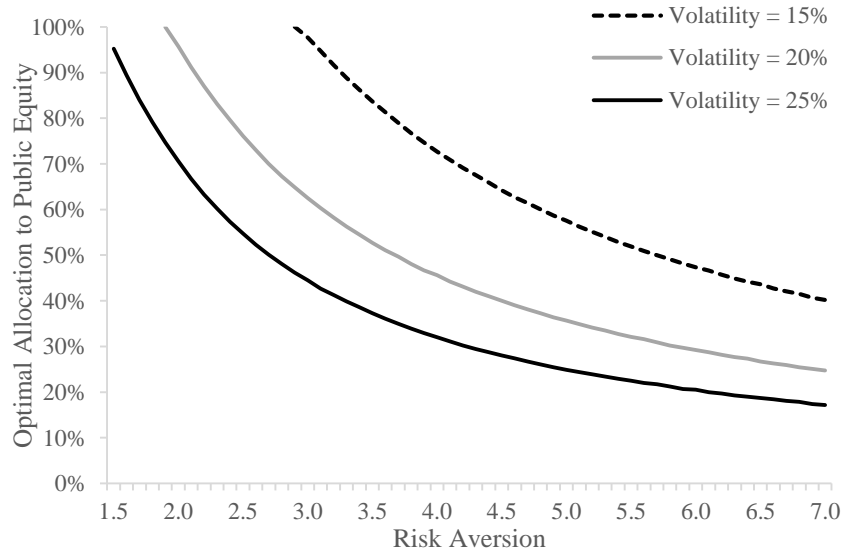


Table 1. Base case Parameter Values

| Inputs | Value | Description |
|-------------------------|--------------|---|
| γ | 1.5-3.5 | LP risk aversion |
| T | 48 | Life of private equity fund in quarters |
| r_f | 4% | Risk-free rate |
| m | 2% | Annual management fee as a percentage of committed capital |
| h | 8% | Hurdle rate expressed as a periodic return on all investment capital and management fees paid |
| c | 20% | Carried interest paid to general partner as a percentage of fund profits after all debt is repaid and hurdle rate is achieved |
| μ_U, σ_U | 10%, 15% | Mean and volatility of public equity returns |
| σ_V | 45% | Volatility of returns for venture capital funds |
| σ_B | 30% | Volatility of returns for buyout funds |
| ρ_V | 0.60 | Correlation between public equity and venture capital returns |
| ρ_B | 0.30 | Correlation between public equity and buyout returns |
| Other parameters | | |
| W_0 | | Initial LP wealth |
| D_0 | | Dollar amount of zero-coupon debt issued by GP to increase asset purchase |
| x, y, z | | Fractions of LP wealth initially allocated to public equity, private equity and risk-free bond |
| X_0, Y_0, Z_0 | | Dollar amount of LP wealth initially allocated to public equity, private equity, and risk-free bond |
| π | | Probability of capital calls |
| ϕ | | Probability of asset sales |
| H^H, H^L | | Multiplicative haircuts for LP exit on secondary market |
| k | | Date of capital call |
| ω | | Probability of a liquidity shock |
| Outputs | | |
| u, v | | Return increments for public and private equity possible at each node in the lattice |
| p_1, p_0, p_{-1} | | Trinomial branch to approximate a normal distribution |
| $p_{i,j}$ | | Branch probabilities to approximate a bivariate normal distribution |
| I_0 | | Dollar amount of committed capital invested by GP |
| A_0 | | Sum of investment capital and debt used for asset purchase |
| A_1, A_2, A_3 | | Asset values at disbursement date defining payoffs to debtholders, LP, and GP |
| $LP(A_t)$ | | LP's payoff |

Table 2. Impact of Uncertainty in Timing of Calls and Distributions

Listed are summary statistics of private equity funds from the perspective of a risk averse investor with a 12-year horizon and risk aversion of 1.5. The five columns correspond to five assumptions about the timing of cash flows in the fund. For calls: the first assumes an immediate call; the second and fourth assume a call in 2.5 years; the third and fifth assume a random call with equal probability each quarter for the first 4.75 years. For distributions: the first assumes a fixed distribution date in 12 years; the fourth and fifth allow for early distributions by the GP; the second and third assume a fixed distribution date set equal to the expected distribution date of the fourth to the nearest quarter. Within each panel asset expected returns, listed in panel headings, are set such that the LP's subjective valuation of a portfolio of 20% risk-free bonds and 80% private equity in the first column equals the subjective valuation of a portfolio of 20% risk-free bonds and 80% public equity. Public equity has 10% expected return and 15% volatility. The risk-free rate is 4%. Panel A models a venture capital fund with no leverage and 45% asset volatility. Panels B and C model a buyout fund with 30% asset volatility and leverage of 1X and 3X equity, respectively. Listed is the certainty equivalent of the portfolios, the expected horizon of capital deployed by the GP, the expected horizon of the commitment between time 0 and distribution, and the probability that an LP will receive a payoff less than the initial commitment.

| | No Early Distribution | | | Early Distribution | |
|---|-----------------------|------------|-------------|--------------------|-------------|
| | Call at 0 | Fixed Call | Random Call | Fixed Call | Random Call |
| Panel A. Venture Capital ($\mu = 20.6\%$) | | | | | |
| Value | 160.76 | 139.83 | 138.95 | 142.47 | 141.64 |
| E[Horizon] | 12.00 | 8.75 | 8.75 | 8.84 | 8.84 |
| E[Commitment] | 12.00 | 11.25 | 11.25 | 11.34 | 11.34 |
| Pr[TVPI<1] | 6.8% | 10.7% | 10.0% | 9.4% | 8.8% |
| Panel B. 1X Buyout ($\mu = 17.3\%$) | | | | | |
| Value | 160.77 | 128.64 | 127.48 | 140.33 | 138.95 |
| E[Horizon] | 12.00 | 7.25 | 7.25 | 7.17 | 7.24 |
| E[Commitment] | 12.00 | 9.75 | 9.75 | 9.67 | 9.74 |
| Pr[TVPI<1] | 4.5% | 7.8% | 9.4% | 6.6% | 6.1% |
| Panel C. 3X Buyout ($\mu = 18.4\%$) | | | | | |
| Value | 160.76 | 124.91 | 125.01 | 155.02 | 151.81 |
| E[Horizon] | 12.00 | 7.00 | 7.00 | 7.02 | 7.09 |
| E[Commitment] | 12.00 | 9.50 | 9.50 | 9.52 | 9.59 |
| Pr[TVPI<1] | 5.1% | 10.7% | 9.3% | 5.3% | 5.6% |

Table 3. Break-even Venture Capital Fund Performance

Listed are the annual break-even expected returns and alphas of venture capital fund assets, in Panels A and B, the corresponding returns for a risk averse investor in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the investor’s subjective valuation of a portfolio of 20% risk-free bonds, $x\%$ venture capital fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% risk-free bonds and the remainder in public equity. The allocation x is varied in the table. Public equity is assumed to have returns with annual mean of 10% and volatility of 15%. The venture capital fund has a maximum 12-year life, no leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns have annual volatility of 45% and correlation with public equity of 0.60. The risk-free rate is 4%. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor’s risk aversion (γ), as well as the multiplicative haircut for venture capital funds which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|----------|--------------------------------------|------------|------------|------------|-------------------------------|------------|------------|------------|
| | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ |
| γ | Panel A. Break-even VC Asset Returns | | | | | | | |
| 1.5 | 14.3% | 16.3% | 18.3% | 20.4% | 12.6% | 14.3% | 16.1% | 18.0% |
| 2.5 | 17.9% | 21.4% | 25.1% | 29.4% | 15.1% | 18.0% | 20.8% | 24.0% |
| 3.5 | 21.2% | 26.5% | 32.7% | 41.9% | 17.2% | 21.0% | 24.9% | 29.4% |
| γ | Panel B. Break-even VC Asset Alphas | | | | | | | |
| 1.5 | -0.5% | 1.5% | 3.5% | 5.6% | -2.2% | -0.5% | 1.3% | 3.2% |
| 2.5 | 3.1% | 6.6% | 10.3% | 14.6% | 0.3% | 3.2% | 6.0% | 9.2% |
| 3.5 | 6.4% | 11.7% | 17.9% | 27.1% | 2.4% | 6.2% | 10.1% | 14.6% |
| γ | Panel C. Break-even VC Fund Returns | | | | | | | |
| 1.5 | 10.0% | 10.4% | 12.6% | 15.0% | 8.0% | 10.0% | 10.3% | 12.5% |
| 2.5 | 12.5% | 15.1% | 17.8% | 20.8% | 10.1% | 12.5% | 15.0% | 17.7% |
| 3.5 | 15.1% | 19.7% | 23.7% | 33.8% | 12.4% | 15.0% | 17.8% | 20.8% |
| γ | Panel D. Break-even VC TVPI | | | | | | | |
| 1.5 | 3.25 | 3.36 | 4.36 | 4.53 | 2.56 | 3.25 | 3.35 | 4.35 |
| 2.5 | 4.34 | 5.86 | 8.02 | 11.25 | 3.28 | 4.35 | 4.61 | 6.10 |
| 3.5 | 5.85 | 8.17 | 15.84 | 35.31 | 3.45 | 5.85 | 8.00 | 11.25 |
| γ | Panel E. Break-even VC PME | | | | | | | |
| 1.5 | 0.95 | 1.13 | 1.35 | 1.63 | 0.84 | 0.95 | 1.12 | 1.31 |
| 2.5 | 1.29 | 1.72 | 2.46 | 3.52 | 1.02 | 1.31 | 1.66 | 2.22 |
| 3.5 | 1.69 | 2.77 | 4.87 | 11.97 | 1.23 | 1.67 | 2.33 | 3.52 |

Table 4. Break-even 1X Buyout Fund Performance

Listed are the annual break-even expected returns and alphas of buyout fund assets, in Panels A and B, the corresponding returns for a risk averse investor in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the investor’s subjective valuation of a portfolio of 20% risk-free bonds, $x\%$ buyout fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% risk-free bonds and the remainder in public equity. The allocation x is varied in the table. Public equity is assumed to have returns with annual mean of 10% and volatility of 15%. The buyout fund has a maximum 12-year life, 1X leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns have annual volatility of 30% and correlation with public equity of 0.30. The risk-free rate is 4%. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor’s risk aversion (γ), as well as the multiplicative haircut for buyout funds which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|----------|--|------------|------------|------------|-------------------------------|------------|------------|------------|
| | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ |
| γ | Panel A. Break-even 1X Buyout Asset Returns | | | | | | | |
| 1.5 | 15.6% | 15.3% | 16.3% | 18.1% | 14.0% | 13.6% | 14.6% | 16.2% |
| 2.5 | 15.9% | 17.4% | 20.1% | 24.3% | 13.6% | 14.7% | 17.0% | 20.6% |
| 3.5 | 16.2% | 19.6% | 24.8% | 33.7% | 13.1% | 15.8% | 19.6% | 25.5% |
| γ | Panel B. Break-even 1X Buyout Asset Alphas | | | | | | | |
| 1.5 | 8.0% | 7.7% | 8.7% | 10.5% | 6.4% | 6.0% | 7.0% | 8.6% |
| 2.5 | 8.3% | 9.8% | 12.5% | 16.7% | 6.0% | 7.1% | 9.4% | 13.0% |
| 3.5 | 8.6% | 12.0% | 17.2% | 26.1% | 5.5% | 8.2% | 12.0% | 17.9% |
| γ | Panel C. Break-even 1X Buyout Fund Returns | | | | | | | |
| 1.5 | 14.2% | 14.0% | 14.5% | 16.2% | 12.7% | 12.4% | 13.5% | 14.4% |
| 2.5 | 14.3% | 15.8% | 17.9% | 20.8% | 12.4% | 13.6% | 15.3% | 18.2% |
| 3.5 | 14.4% | 17.3% | 21.3% | 28.5% | 12.2% | 14.2% | 17.3% | 22.2% |
| γ | Panel D. Break-even 1X Buyout TVPI | | | | | | | |
| 1.5 | 3.52 | 3.46 | 3.65 | 4.25 | 3.07 | 3.01 | 3.31 | 3.62 |
| 2.5 | 3.58 | 3.81 | 4.66 | 6.09 | 3.01 | 3.34 | 3.75 | 4.74 |
| 3.5 | 3.62 | 4.59 | 6.18 | 13.45 | 2.93 | 3.56 | 4.59 | 7.31 |
| γ | Panel E. Break-even 1X Buyout PME | | | | | | | |
| 1.5 | 1.40 | 1.38 | 1.48 | 1.67 | 1.25 | 1.22 | 1.31 | 1.46 |
| 2.5 | 1.43 | 1.58 | 1.91 | 2.53 | 1.22 | 1.32 | 1.55 | 1.97 |
| 3.5 | 1.46 | 1.84 | 2.61 | 5.00 | 1.18 | 1.42 | 1.84 | 2.75 |

Table 5. Break-even 3X Buyout Fund Performance

Listed are the annual break-even expected returns and alphas of buyout fund assets, in Panels A and B, the corresponding returns for a risk averse investor in Panel C, as well as the corresponding total values to paid in capital (TVPI) and public market equivalents (PME) in Panels D and E, respectively. The break-even asset expected return is determined by setting the investor's subjective valuation of a portfolio of 20% risk-free bonds, $x\%$ buyout fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% risk-free bonds and the remainder in public equity. The allocation x is varied in the table. Public equity is assumed to have returns with annual mean of 10% and volatility of 15%. The buyout fund has a maximum 12-year life, 3X leverage, 2% annual management fee, and a 20% performance fee paid above a hurdle rate of 8%. Fund asset returns have annual volatility of 30% and correlation with public equity of 0.30. The risk-free rate is 4%. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor's risk aversion (γ), as well as the multiplicative haircut for buyout funds which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|----------|---|------------|------------|------------|-------------------------------|------------|------------|------------|
| | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ |
| γ | Panel A. Break-even 3X Buyout Asset Returns | | | | | | | |
| 1.5 | 10.1% | 11.0% | 12.7% | 15.4% | 9.0% | 9.8% | 11.5% | 14.2% |
| 2.5 | 11.5% | 14.2% | 18.0% | 23.9% | 9.9% | 12.4% | 15.9% | 21.4% |
| 3.5 | 12.9% | 17.7% | 24.4% | 38.2% | 10.7% | 15.1% | 20.7% | 30.7% |
| γ | Panel B. Break-even 3X Buyout Asset Alphas | | | | | | | |
| 1.5 | 2.5% | 3.4% | 5.1% | 7.8% | 1.4% | 2.2% | 3.9% | 6.6% |
| 2.5 | 3.9% | 6.6% | 10.4% | 16.3% | 2.3% | 4.8% | 8.3% | 13.8% |
| 3.5 | 5.3% | 10.1% | 16.8% | 30.6% | 3.1% | 7.5% | 13.1% | 23.1% |
| γ | Panel C. Break-even 3X Buyout Fund Returns | | | | | | | |
| 1.5 | 14.1% | 14.7% | 16.7% | 19.3% | 12.5% | 13.9% | 15.3% | 18.6% |
| 2.5 | 15.3% | 18.6% | 21.7% | 27.6% | 13.9% | 16.6% | 19.8% | 25.4% |
| 3.5 | 16.8% | 21.5% | 28.1% | 39.9% | 14.5% | 19.1% | 24.4% | 33.4% |
| γ | Panel D. Break-even 3X Buyout TVPI | | | | | | | |
| 1.5 | 3.50 | 3.79 | 4.37 | 5.58 | 3.03 | 3.35 | 3.91 | 4.96 |
| 2.5 | 3.91 | 4.96 | 6.62 | 10.80 | 3.39 | 4.25 | 5.80 | 8.54 |
| 3.5 | 4.46 | 6.48 | 10.99 | 36.59 | 3.74 | 5.45 | 8.28 | 19.14 |
| γ | Panel E. Break-even 3X Buyout PME | | | | | | | |
| 1.5 | 1.34 | 1.47 | 1.73 | 2.19 | 1.19 | 1.30 | 1.54 | 1.97 |
| 2.5 | 1.54 | 1.97 | 2.71 | 4.32 | 1.32 | 1.68 | 2.28 | 3.55 |
| 3.5 | 1.76 | 2.64 | 4.48 | 13.75 | 1.43 | 2.13 | 3.36 | 7.37 |

Table 6. Reported versus Realized Performance for Venture Capital Funds

Listed are break-even private equity fund returns and PMEs. These are determined by first finding the expected asset return that sets the investor’s subjective valuation of a portfolio of 20% risk-free bonds, $x\%$ private equity fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% risk-free bonds and the remainder in public equity. The allocation x is varied in the table. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor’s risk aversion (γ), as well as the multiplicative haircut levied on private equity secondary market transactions which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity. Given the break-even asset return, the corresponding break-even fund return is the median return taking into account possible realizations for the LP. The “Reported” performance measures only incorporate realizations from distributions by the GP. The “Realized” performance measures also incorporate realizations from secondary market transactions which are triggered by rare liquidity shocks.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|----------|--|------------|------------|------------|-------------------------------|------------|------------|------------|
| | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ |
| γ | Panel A. Break-even Reported VC Fund Returns | | | | | | | |
| 1.5 | 10.0% | 10.4% | 12.6% | 15.0% | 8.0% | 10.0% | 10.3% | 12.5% |
| 2.5 | 12.5% | 15.1% | 17.8% | 20.8% | 10.1% | 12.5% | 15.0% | 17.7% |
| 3.5 | 15.1% | 19.7% | 23.7% | 33.8% | 12.4% | 15.0% | 17.8% | 20.8% |
| γ | Panel B. Break-even Realized VC Fund Returns | | | | | | | |
| 1.5 | 6.4% | 7.7% | 8.1% | 10.1% | 6.5% | 7.7% | 8.1% | 10.0% |
| 2.5 | 8.0% | 10.4% | 12.9% | 16.7% | 7.8% | 10.0% | 12.3% | 14.2% |
| 3.5 | 10.3% | 15.1% | 18.8% | 27.4% | 9.6% | 12.4% | 15.1% | 18.0% |
| γ | Panel C. Break-even Reported VC PME | | | | | | | |
| 1.5 | 0.95 | 1.13 | 1.35 | 1.63 | 0.84 | 0.95 | 1.12 | 1.31 |
| 2.5 | 1.29 | 1.72 | 2.46 | 3.52 | 1.02 | 1.31 | 1.66 | 2.22 |
| 3.5 | 1.69 | 2.77 | 4.87 | 11.97 | 1.23 | 1.67 | 2.33 | 3.52 |
| γ | Panel D. Break-even Realized VC PME | | | | | | | |
| 1.5 | 0.75 | 0.83 | 0.92 | 1.03 | 0.83 | 0.89 | 0.96 | 1.04 |
| 2.5 | 0.90 | 1.11 | 1.39 | 1.90 | 0.92 | 1.04 | 1.20 | 1.44 |
| 3.5 | 1.10 | 1.54 | 2.40 | 5.09 | 1.01 | 1.22 | 1.53 | 2.08 |

Table 7. Reported versus Realized Performance for 1X Buyout Funds

Listed are break-even private equity fund returns and PMEs. These are determined by first finding the expected asset return that sets the investor’s subjective valuation of a portfolio of 20% risk-free bonds, $x\%$ private equity fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% risk-free bonds and the remainder in public equity. The allocation x is varied in the table. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor’s risk aversion (γ), as well as the multiplicative haircut levied on private equity secondary market transactions which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity. Given the break-even asset return, the corresponding break-even fund return is the median return taking into account possible realizations for the LP. The “Reported” performance measures only incorporate realizations from distributions by the GP. The “Realized” performance measures also incorporate realizations from secondary market transactions which are triggered by rare liquidity shocks.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|----------|---|------------|------------|------------|-------------------------------|------------|------------|------------|
| | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ |
| γ | Panel C. Break-even Reported 1X Buyout Fund Returns | | | | | | | |
| 1.5 | 14.2% | 14.0% | 14.5% | 16.2% | 12.7% | 12.4% | 13.5% | 14.4% |
| 2.5 | 14.3% | 15.8% | 17.9% | 20.8% | 12.4% | 13.6% | 15.3% | 18.2% |
| 3.5 | 14.4% | 17.3% | 21.3% | 28.5% | 12.2% | 14.2% | 17.3% | 22.2% |
| γ | Panel B. Break-even Realized 1X Buyout Fund Returns | | | | | | | |
| 1.5 | 10.8% | 10.6% | 11.4% | 12.7% | 10.6% | 10.4% | 11.1% | 12.4% |
| 2.5 | 11.0% | 12.3% | 14.4% | 17.3% | 10.4% | 11.2% | 12.8% | 15.8% |
| 3.5 | 11.3% | 14.1% | 17.8% | 24.4% | 10.1% | 12.2% | 14.8% | 19.2% |
| γ | Panel C. Break-even Reported 1X Buyout PME | | | | | | | |
| 1.5 | 1.40 | 1.38 | 1.48 | 1.67 | 1.25 | 1.22 | 1.31 | 1.46 |
| 2.5 | 1.43 | 1.58 | 1.91 | 2.53 | 1.22 | 1.32 | 1.55 | 1.97 |
| 3.5 | 1.46 | 1.84 | 2.61 | 5.00 | 1.18 | 1.42 | 1.84 | 2.75 |
| γ | Panel D. Break-even Realized 1X Buyout PME | | | | | | | |
| 1.5 | 1.04 | 1.02 | 1.09 | 1.20 | 1.06 | 1.04 | 1.09 | 1.18 |
| 2.5 | 1.06 | 1.16 | 1.35 | 1.70 | 1.04 | 1.10 | 1.23 | 1.48 |
| 3.5 | 1.08 | 1.31 | 1.74 | 2.89 | 1.02 | 1.16 | 1.41 | 1.94 |

Table 8. Reported versus Realized Performance for 3X Buyout Funds

Listed are break-even private equity fund returns and PMEs. These are determined by first finding the expected asset return that sets the investor’s subjective valuation of a portfolio of 20% risk-free bonds, $x\%$ private equity fund, and the remainder in public equity equal to the subjective valuation of a portfolio of 20% risk-free bonds and the remainder in public equity. The allocation x is varied in the table. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor’s risk aversion (γ), as well as the multiplicative haircut levied on private equity secondary market transactions which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity. Given the break-even asset return, the corresponding break-even fund return is the median return taking into account possible realizations for the LP. The “Reported” performance measures only incorporate realizations from distributions by the GP. The “Realized” performance measures also incorporate realizations from secondary market transactions which are triggered by rare liquidity shocks.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|----------|---|------------|------------|------------|-------------------------------|------------|------------|------------|
| | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ | $x = 20\%$ | $x = 40\%$ | $x = 60\%$ | $x = 80\%$ |
| γ | Panel C. Break-even Reported 3X Buyout Fund Returns | | | | | | | |
| 1.5 | 14.1% | 14.7% | 16.7% | 19.3% | 12.5% | 13.9% | 15.3% | 18.6% |
| 2.5 | 15.3% | 18.6% | 21.7% | 27.6% | 13.9% | 16.6% | 19.8% | 25.4% |
| 3.5 | 16.8% | 21.5% | 28.1% | 39.9% | 14.5% | 19.1% | 24.4% | 33.4% |
| γ | Panel B. Break-even Realized 3X Buyout Fund Returns | | | | | | | |
| 1.5 | 9.9% | 11.3% | 12.9% | 15.9% | 9.8% | 11.1% | 12.7% | 15.8% |
| 2.5 | 11.7% | 14.6% | 18.6% | 24.0% | 11.2% | 14.0% | 17.3% | 22.7% |
| 3.5 | 13.2% | 18.1% | 24.3% | 36.5% | 11.9% | 16.7% | 21.9% | 31.4% |
| γ | Panel C. Break-even Reported 3X Buyout PME | | | | | | | |
| 1.5 | 1.34 | 1.47 | 1.73 | 2.19 | 1.19 | 1.30 | 1.54 | 1.97 |
| 2.5 | 1.54 | 1.97 | 2.71 | 4.32 | 1.32 | 1.68 | 2.28 | 3.55 |
| 3.5 | 1.76 | 2.64 | 4.48 | 13.75 | 1.43 | 2.13 | 3.36 | 7.37 |
| γ | Panel D. Break-even Realized 3X Buyout PME | | | | | | | |
| 1.5 | 0.96 | 1.04 | 1.20 | 1.50 | 1.00 | 1.06 | 1.20 | 1.48 |
| 2.5 | 1.08 | 1.36 | 1.81 | 2.72 | 1.07 | 1.29 | 1.67 | 2.46 |
| 3.5 | 1.22 | 1.78 | 2.81 | 7.02 | 1.13 | 1.58 | 2.35 | 4.55 |

Table 9. Impact of Fees

Listed are break-even private equity asset returns as a function of the annual management fee and carried interest on private equity fund profits paid above a hurdle rate of 8%. The break-even returns are determined by setting an investor's subjective valuation of a portfolio of 20% risk-free bonds earning 4% and 80% public equity equal to the valuation of a portfolio of 20% risk-free bonds, 40% public equity, and 40% private equity. Public equity has expected return of 10% and volatility of 15%. The investor has risk aversion of 2.5. In Panel A, the venture capital fund has a maximum 12-year life, annual asset return volatility of 45%, asset correlation of 0.60 with public equity, and no leverage. In Panel B, the 1X buyout fund has a maximum 12-year life, annual asset return volatility of 30%, asset correlation of 0.30 with public equity, and 100% leverage. In Panel C, the 3X buyout fund is identical to that in Panel B save for 300% leverage. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Two sets of haircuts are considered.

| | $H^L = 80\%$ and $H^H = 50\%$ | | | | $H^L = 95\%$ and $H^H = 90\%$ | | | |
|---------|-------------------------------|-------|-------|-------|-------------------------------|-------|-------|-------|
| | Panel A. Venture Capital | | | | | | | |
| | Carried Interest | | | | Carried Interest | | | |
| Mgt Fee | 0% | 15% | 20% | 25% | 0% | 15% | 20% | 25% |
| 0.0% | 17.2% | 18.9% | 19.3% | 19.7% | 14.0% | 15.4% | 15.7% | 16.1% |
| 1.5% | 19.8% | 21.4% | 21.8% | 22.2% | 16.6% | 18.0% | 18.3% | 18.6% |
| 2.0% | 20.5% | 22.1% | 22.5% | 22.9% | 17.3% | 18.7% | 19.0% | 19.4% |
| 2.5% | 21.3% | 22.9% | 23.3% | 23.7% | 18.1% | 19.4% | 19.8% | 20.1% |
| | Panel B. 1X Buyout | | | | | | | |
| | Carried Interest | | | | Carried Interest | | | |
| Mgt Fee | 0% | 15% | 20% | 25% | 0% | 15% | 20% | 25% |
| 0.0% | 12.5% | 15.4% | 15.9% | 16.4% | 10.5% | 12.9% | 13.2% | 13.6% |
| 1.5% | 14.2% | 17.4% | 17.8% | 18.3% | 12.1% | 14.7% | 15.1% | 15.5% |
| 2.0% | 14.7% | 17.9% | 18.4% | 18.9% | 12.6% | 15.2% | 15.6% | 16.0% |
| 2.5% | 15.3% | 18.5% | 18.9% | 19.4% | 13.1% | 15.7% | 16.1% | 16.5% |
| | Panel C. 3X Buyout | | | | | | | |
| | Carried Interest | | | | Carried Interest | | | |
| Mgt Fee | 0% | 15% | 20% | 25% | 0% | 15% | 20% | 25% |
| 0.0% | 11.3% | 12.7% | 13.0% | 13.4% | 9.8% | 11.0% | 11.3% | 11.6% |
| 1.5% | 12.5% | 14.2% | 14.5% | 14.9% | 10.9% | 12.4% | 12.7% | 13.0% |
| 2.0% | 12.8% | 14.6% | 14.9% | 15.3% | 11.3% | 12.8% | 13.1% | 13.4% |
| 2.5% | 13.2% | 15.1% | 15.4% | 15.7% | 11.6% | 13.2% | 13.5% | 13.8% |

Table 10. Optimal Portfolio Weights

Listed are utility-maximizing portfolio weights for public equity and private equity from the perspective of a risk averse investor with a 12-year horizon. In all cases allocation to a 4% risk-free bond is fixed at 20%. The remaining 80% is split across public and private equity. Public equity has expected return of 10% and volatility of 15%. In Panel A, venture capital assets have expected return of 18%, volatility of 45%, and correlation of 0.60 with public equity. In Panel B, 1X buyout fund assets have expected return of 15%, volatility of 30%, and correlation of 0.30 with public equity. In Panel C, 3X Buyout fund assets have expected return of 12%, volatility of 30%, and correlation of 0.30 with public equity. In all cases the investor is forced to sell risky assets on a secondary market during liquidity shocks occurring at a base rate of 1% per quarter which rises 1% for each quarter that public equity declines. Public equity is sold at fair value whereas private equity is sold with a multiplicative haircut applied to fair value. Results vary within the panels with the investor's risk aversion (γ), as well as with the multiplicative haircut levied on secondary market transactions which is either low (H^L) or high (H^H). The probability of a high haircut is inversely related to the cumulative performance of public equity.

| Panel A. Venture Capital | | | | |
|--------------------------|-------------------------------|------|-------------------------------|------|
| γ | $H^L = 80\%$ and $H^H = 50\%$ | | $H^L = 95\%$ and $H^H = 90\%$ | |
| | Public | VC | Public | VC |
| 1.5 | 0.55 | 0.25 | 0.43 | 0.37 |
| 2.5 | 0.70 | 0.10 | 0.62 | 0.18 |
| 3.5 | 0.75 | 0.05 | 0.69 | 0.11 |

| Panel B. 1X Buyout | | | | |
|--------------------|-------------------------------|--------|-------------------------------|--------|
| γ | $H^L = 80\%$ and $H^H = 50\%$ | | $H^L = 95\%$ and $H^H = 90\%$ | |
| | Public | Buyout | Public | Buyout |
| 1.5 | 0.50 | 0.30 | 0.39 | 0.41 |
| 2.5 | 0.62 | 0.18 | 0.53 | 0.27 |
| 3.5 | 0.67 | 0.13 | 0.60 | 0.20 |

| Panel C. 3X Buyout | | | | |
|--------------------|-------------------------------|--------|-------------------------------|--------|
| γ | $H^L = 80\%$ and $H^H = 50\%$ | | $H^L = 95\%$ and $H^H = 90\%$ | |
| | Public | Buyout | Public | Buyout |
| 1.5 | 0.50 | 0.30 | 0.44 | 0.36 |
| 2.5 | 0.64 | 0.16 | 0.59 | 0.21 |
| 3.5 | 0.69 | 0.11 | 0.66 | 0.14 |